Applications of Numerical Homotopy Continuation to Mechanism Design

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November 13, 2018

Nonlinear Algebra in Applications

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Berkeley
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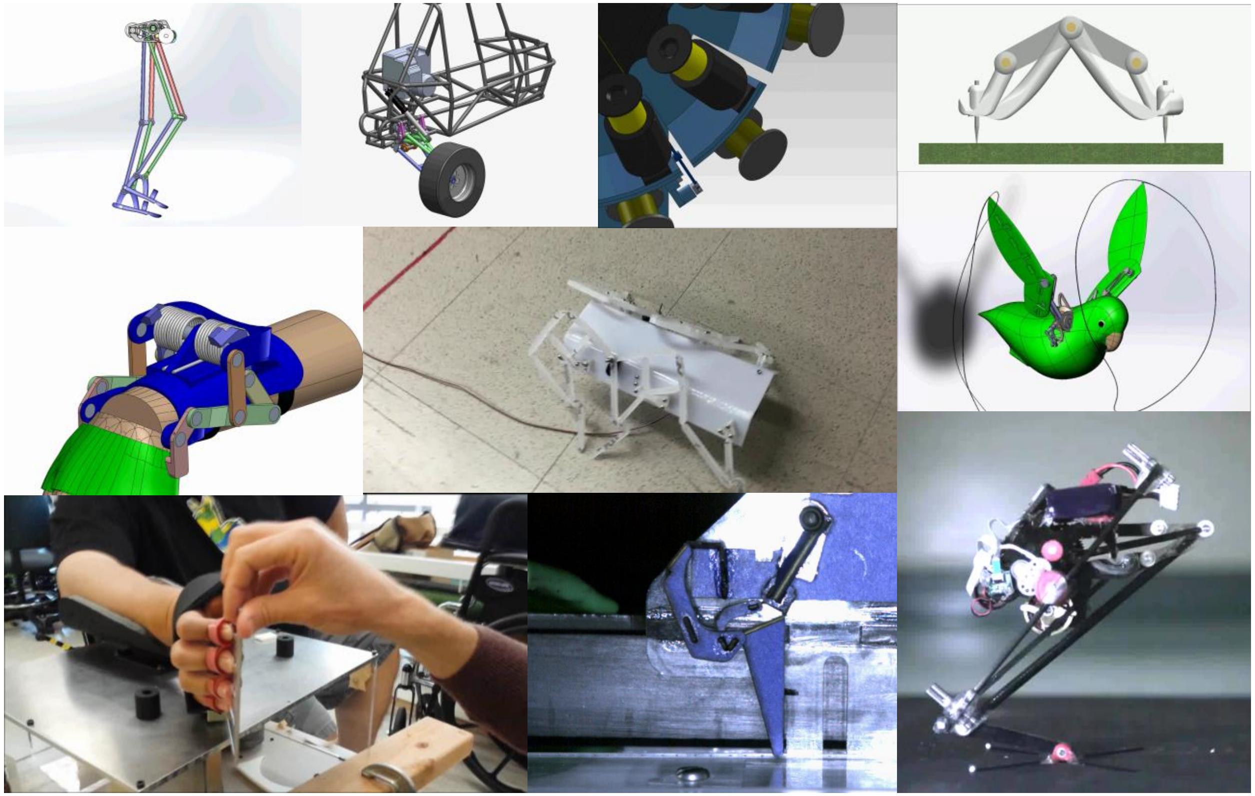
Motivation

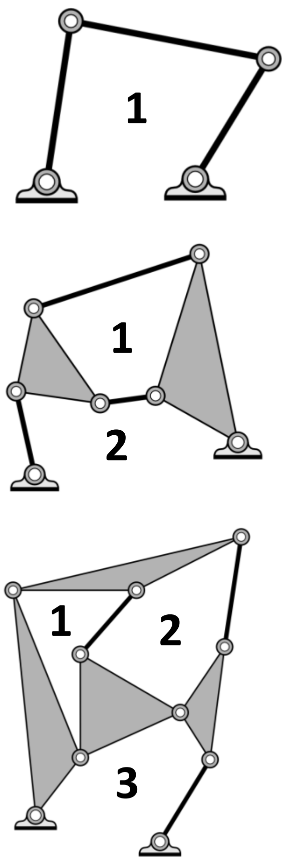
Inventing machines through computation…



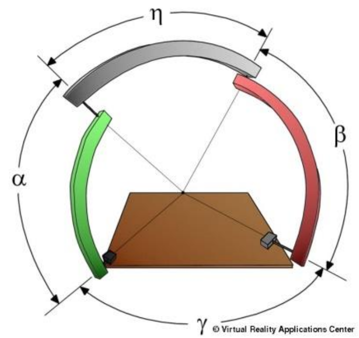
Central Design Element

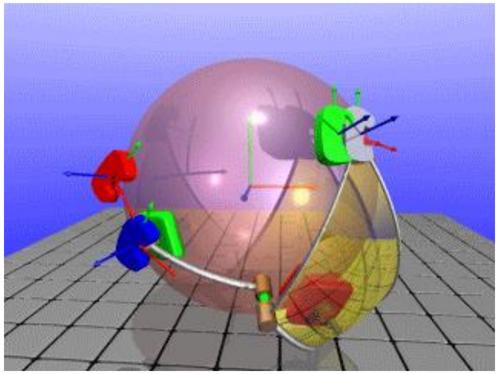
Linkages:

Planar

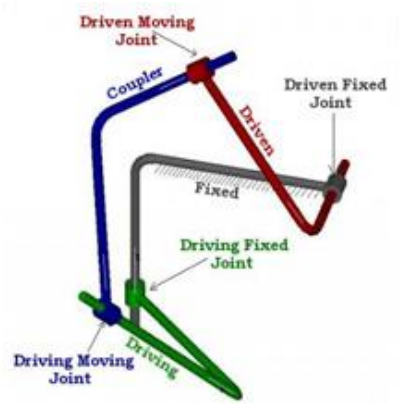


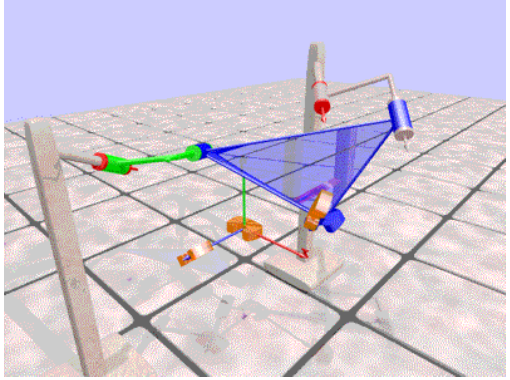
Spherical





Spatial



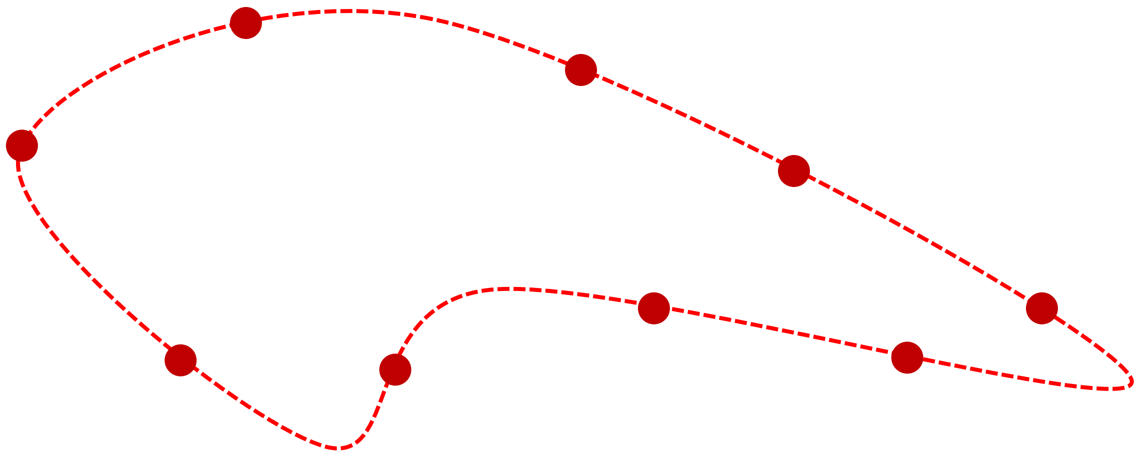


Images courtesy UC Irvine Robotics & Automation Laboratory

Typical Problem Statement

Trace a plane curve:

One approach:
Break curve into discrete points



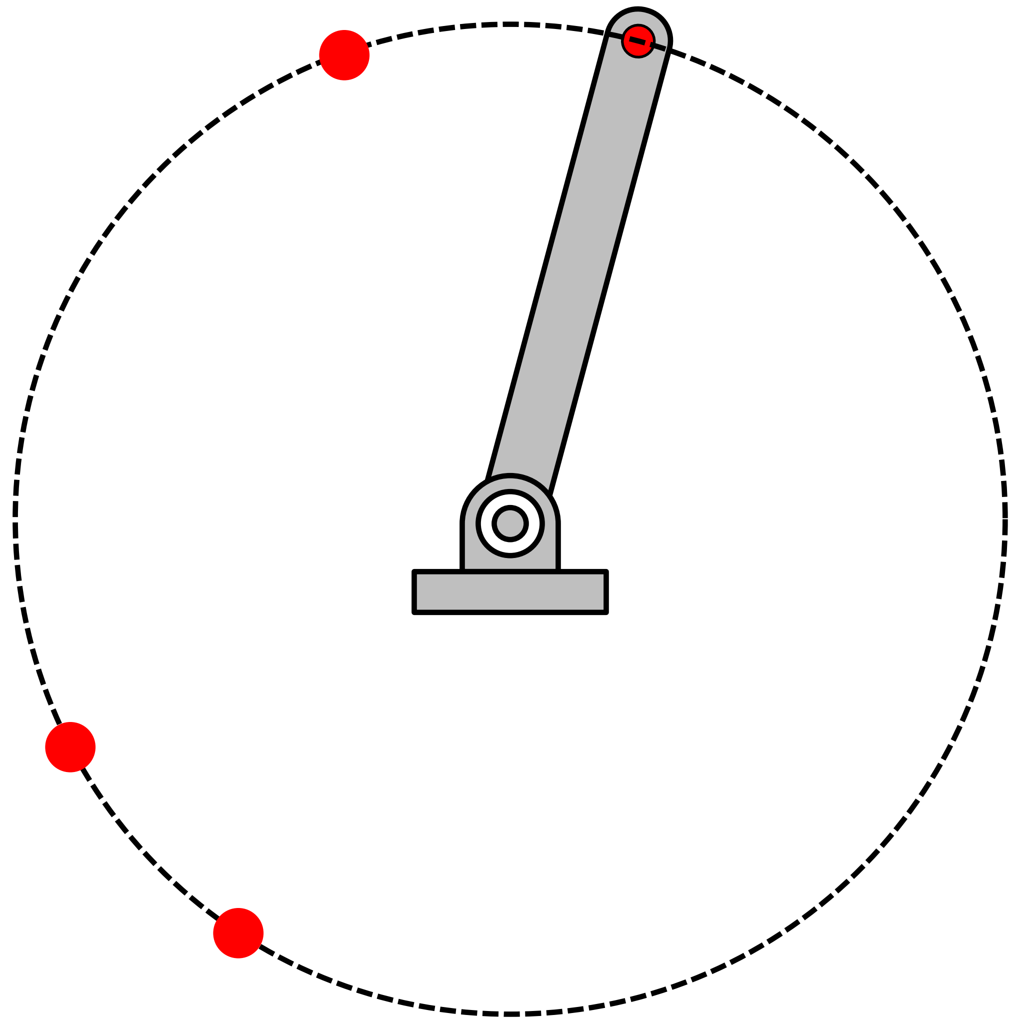
http://www.partsw.com

A Simplified History

The simplest linkage:

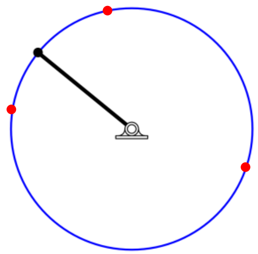
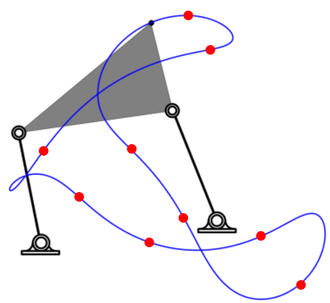
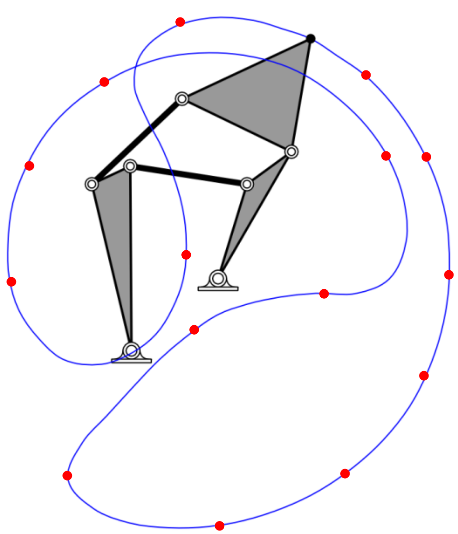
A crank…

…can go through 3 points



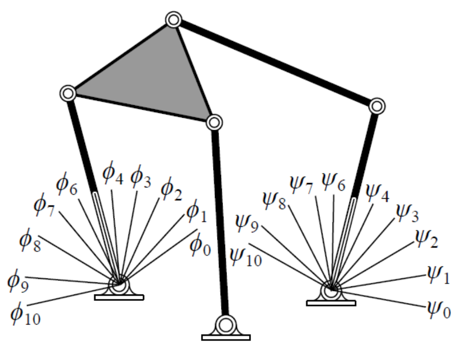
Result date: Unknown

A Simplified History

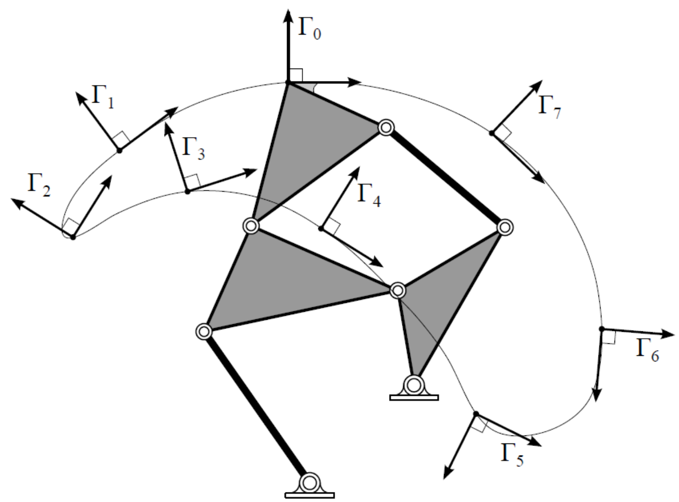
		No. of Points	No. of Mechanisms	
Crank		3	1	(first discovered in ?)
Four-bar		9	4,326	(first computed in 1992)
Six-bar		15	Unknown >1,000,000	(not yet known)

Synthesis Objectives

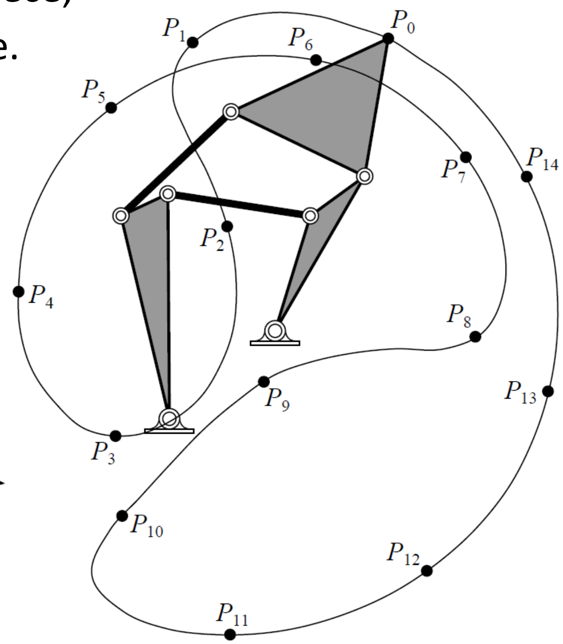
Function generation: set of input angles and output angles;
Motion generation: set of positions and orientations of a workpiece;
Path generation: set of points along a trajectory in the workpiece.



Function Generation

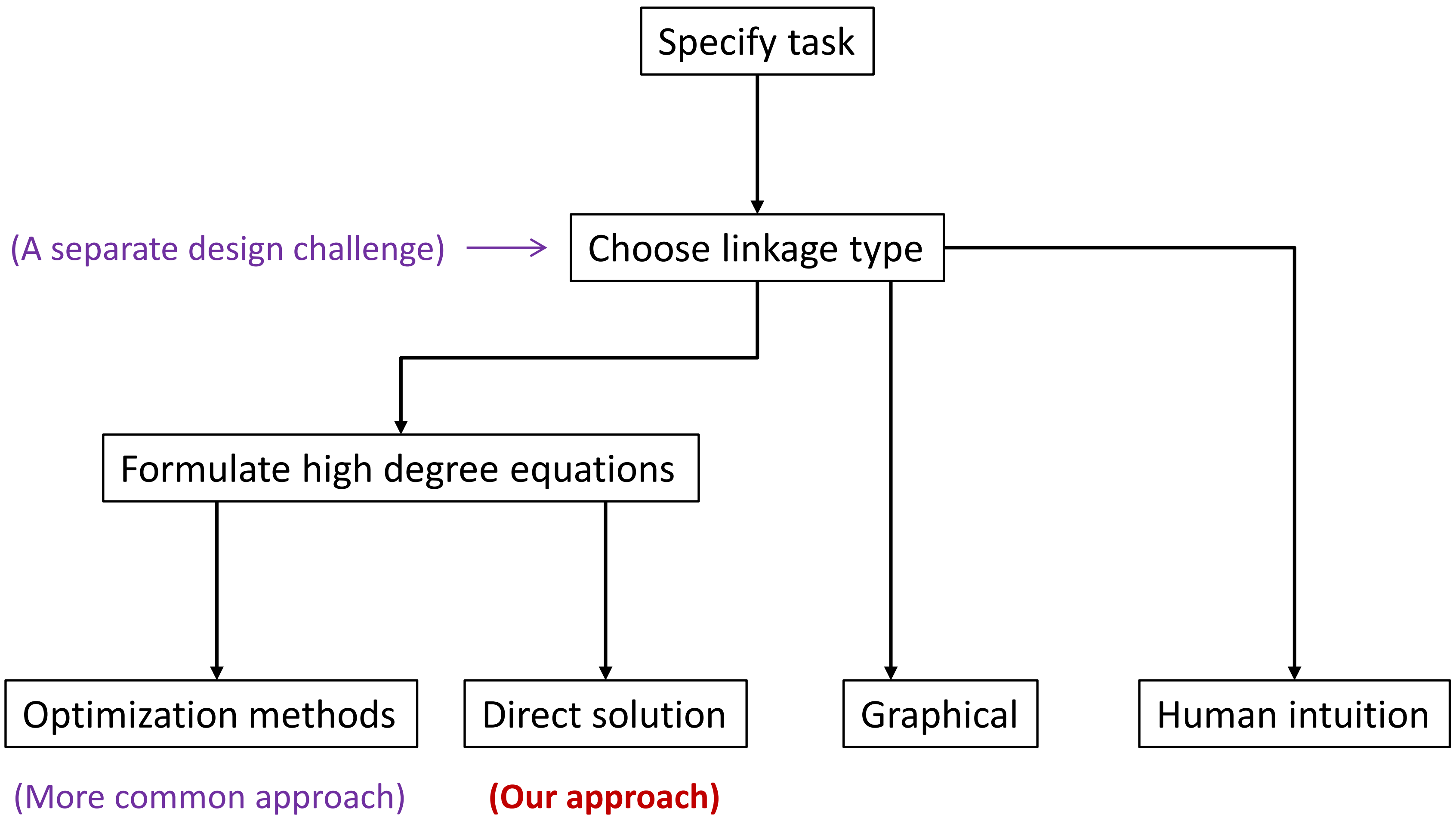


Motion Generation



Path Generation

Synthesis Procedures



Literature Review

[1] **B. Roth and F. Freudenstein, 1963.** "Synthesis of path-generating mechanisms by numerical methods," *J. of Engineering for Industry*, 85(3):298-304.

[2] **A. P. Morgan and A. J. Sommese, 1987.** “A homotopy for solving general polynomial systems that respects m-homogeneous structures,” *Applied Mathematics and Computation*, vol. 24, no. 2, pp. 101–113.

[3] **C. W. Wampler, A. J. Sommese, and A. P. Morgan, 1992.** "Complete solution of the nine-point path synthesis problem for four-bar linkages," *J. of Mech. Des.* 114(1):153-159.

[4] **A. K. Dhingra, J. C. Cheng, and D. Kohli, 1994.** "Synthesis of six-link, slider-crank and four-link mechanisms for function, path and motion generation using homotopy with m-homogenization," *J. of Mech. Des.* 116(4):1122-1131.

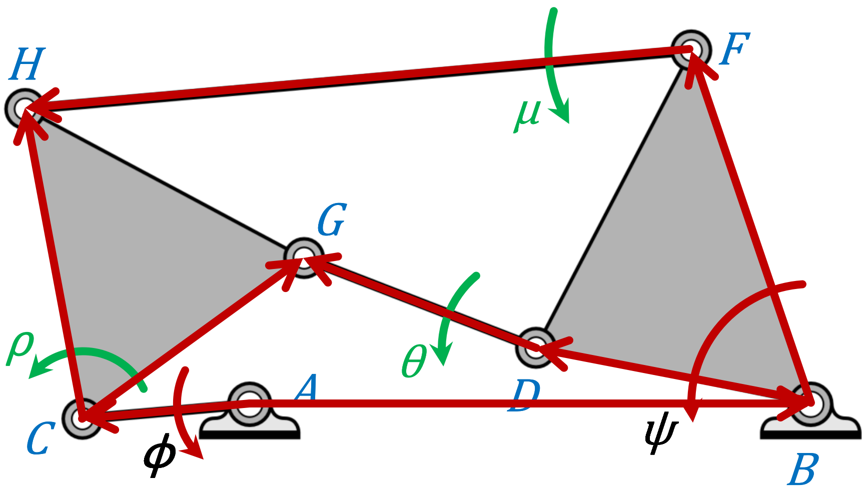
[5] **H.-J. Su, J. M. McCarthy, M. Sosonkina, and L. T. Watson, 2006.** “Algorithm 857: POLSYS GLP—a Parallel General Linear Product Homotopy Code for Solving Polynomial Systems of Equations,” *ACM Trans. Math. Softw*., vol. 32, no. 4, pp. 561–579.

[6] **J. D. Hauenstein, A. J. Sommese, and C. Wampler, 2011.** “Regeneration homotopies for solving systems of polynomials,” *Mathematics of Computation*, vol. 80, no. 273, pp. 345–377.

[7] **D. J. Bates, J. D. Hauenstein, A. J. Sommese and C. W. Wampler, 2013.** *Numerically Solving Polynomial Systems With Bertini*, SIAM Press, Philadelphia, PA, p. 25.

Function Generator

Coordinate input crank with output crank



Stephenson II

Task (specified)

$(0, 0), (\phi_1, \psi_1), (\phi_2, \psi_2), (\phi_3, \psi_3),$
$(\phi_4, \psi_4), (\phi_5, \psi_5), (\phi_6, \psi_6), (\phi_7, \psi_7),$
$(\phi_8, \psi_8), (\phi_9, \psi_9), (\phi_{10}, \psi_{10})$

$Q = e^{i\phi} \quad S = e^{i\psi}$

Joint coordinates (unknowns)

$A \quad B \quad C \quad D \quad F \quad G \quad H$

Rotation operators (extra unknowns)

$R = e^{i\rho} \qquad T = e^{i\theta} \qquad U = e^{i\mu}$

Loop equations (constraints)

$$A + Q_j(C - A) + R_j(G - C) = B + S_j(D - B) + T_j(G - D),$$
$$A + Q_j(C - A) + R_j(H - C) = B + S_j(F - B) + U_j(H - F), \quad j = 1, \dots, N - 1$$

Synthesis Equations

- Loop equations:

$$A+Q_j(C-A)+R_j(G-C)=B+S_j(D-B)+T_j(G-D),$$
$$A+Q_j(C-A)+R_j(H-C)=B+S_j(F-B)+U_j(H-F),$$

- Conjugate loop equations: $j=1,\ldots,N-1$

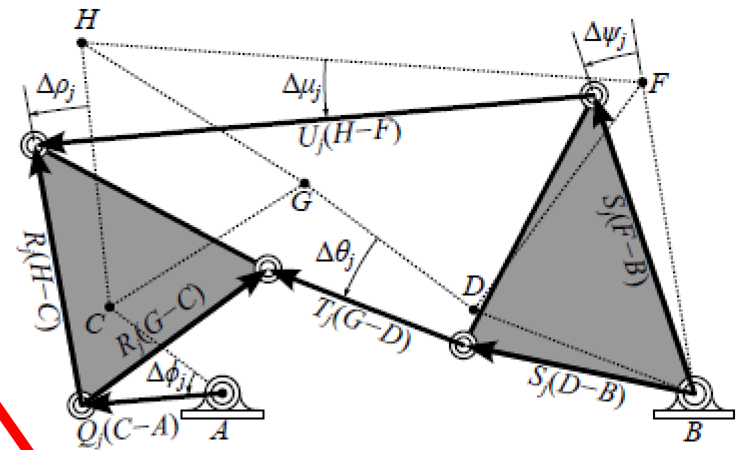
$$\bar{A}+\bar{Q}_j(\bar{C}-\bar{A})+\bar{R}_j(\bar{G}-\bar{C})=\bar{B}+\bar{S}_j(\bar{D}-\bar{B})+\bar{T}_j(\bar{G}-\bar{D}),$$
$$\bar{A}+\bar{Q}_j(\bar{C}-\bar{A})+\bar{R}_j(\bar{H}-\bar{C})=\bar{B}+\bar{S}_j(\bar{F}-\bar{B})+\bar{U}_j(\bar{H}-\bar{F}),$$
$$j=1,\ldots,N-1$$

- Rotation operators:

$$R_j\bar{R}_j=1,\quad T_j\bar{T}_j=1,\quad U_j\bar{U}_j=1,\quad j=1,\ldots N-1$$

- Unknowns:

$$\langle C,\bar{C},D,\bar{D},F,\bar{F},G,\bar{G}\rangle,$$
$$\langle R_j,\bar{R}_j,T_j,\bar{T}_j,U_j,\bar{U}_j\rangle,\quad j=1,\ldots N-1$$



Stephenson II linkage

Synthesis Equations: there are $7(N-1)$

Unknowns: there are $10+6(N-1)$

System square for $N=11$, 70 eqns and unknowns, degree = 1.18×10^{21}

Algebraic Reduction

$$A+Q_j(C-A)+R_j(G-C)=B+S_j(D-B)+T_j(G-D),$$
$$\bar{A}+\bar{Q}_j(\bar{C}-\bar{A})+\bar{R}_j(\bar{G}-\bar{C})=\bar{B}+\bar{S}_j(\bar{D}-\bar{B})+\bar{T}_j(\bar{G}-\bar{D}),$$

$$A+Q_j(C-A)+R_j(H-C)=B+S_j(F-B)+U_j(H-F),$$
$$\bar{A}+\bar{Q}_j(\bar{C}-\bar{A})+\bar{R}_j(\bar{H}-\bar{C})=\bar{B}+\bar{S}_j(\bar{F}-\bar{B})+\bar{U}_j(\bar{H}-\bar{F}),$$

$$T_j\bar{T}_j=1 \qquad U_j\bar{U}_j=1$$

These unknowns are eliminated:

$R_j,\bar{R}_j,\quad T_j,\bar{T}_j,\quad U_j,\bar{U}_j,$

$j=1,\ldots N-1$

$$\begin{bmatrix} a\bar{b}_j & \bar{a}b_j \\ c\bar{d}_j & \bar{c}d_j \end{bmatrix}\begin{Bmatrix} R_j \\ \bar{R}_j \end{Bmatrix}=\begin{Bmatrix} f\bar{f}-a\bar{a}-b_j\bar{b}_j \\ g\bar{g}-c\bar{c}-d_j\bar{d}_j \end{Bmatrix},$$

$$R_j\bar{R}_j=1$$

$$\left(a\bar{b}_j\left(g\bar{g}-c\bar{c}-d_j\bar{d}_j\right)-c\bar{d}_j\left(f\bar{f}-a\bar{a}-b_j\bar{b}_j\right)\right)\left(\bar{a}b_j\left(g\bar{g}-c\bar{c}-d_j\bar{d}_j\right)-\bar{c}d_j\left(f\bar{f}-a\bar{a}-b_j\bar{b}_j\right)\right)+\left(a\bar{b}_j\bar{c}d_j+\bar{a}b_jc\bar{d}_j\right)^2=0$$

$$a=G-C,\quad b_j=A-B+Q_j(C-A)-S_j(D-B),\quad f=G-D,$$
$$c=H-C,\quad d_j=A-B+Q_j(C-A)-S_j(F-B),\quad g=H-F$$

$$j=1,\ldots,10$$

10 synthesis equations in 10 unknowns:

$$\langle C,\bar{C},D,\bar{D},F,\bar{F},G,\bar{G},H,\bar{H}\rangle$$

Degree of the Synthesis Equations

Synthesis equations:

$$\left(a\bar{b}_j\left(g\bar{g}-c\bar{c}-d_j\bar{d}_j\right)-c\bar{d}_j\left(f\bar{f}-a\bar{a}-b_j\bar{b}_j\right)\right)\left(\bar{a}b_j\left(g\bar{g}-c\bar{c}-d_j\bar{d}_j\right)-\bar{c}d_j\left(f\bar{f}-a\bar{a}-b_j\bar{b}_j\right)\right)+\left(a\bar{b}_j\bar{c}d_j+\bar{a}b_jc\bar{d}_j\right)^2=0$$

$$j=1,\ldots,10$$

- Goal: To find all of the solutions $\langle C, \bar{C}, D, \bar{D}, F, \bar{F}, G, \bar{G}, H, \bar{H}\rangle$ of the synthesis equations
- Each polynomial is degree 8
- How many roots?
  - Using Bezout's Theorem:

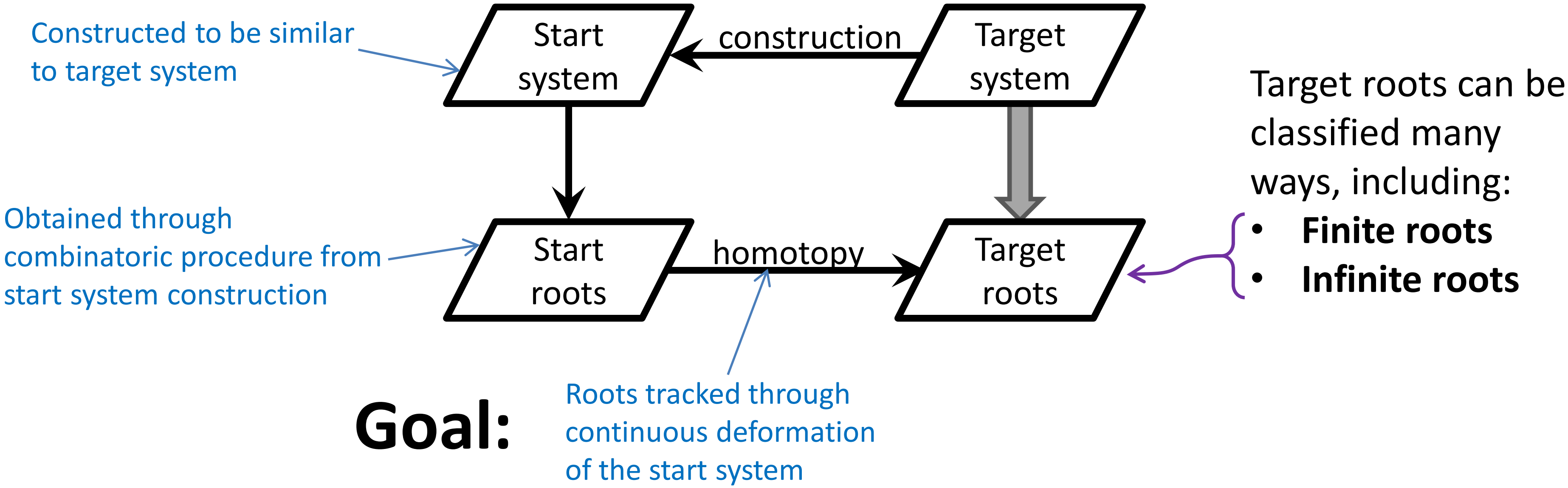
    $8^{10} = 1.07\times10^9$
  - Using a multihomogeneous grouping: $\langle C, D, F, G, H\rangle, \langle \bar{C}, \bar{D}, \bar{F}, \bar{G}, \bar{H}\rangle$

    264,241,152

This is the lowest bound we can compute.
Uses sparse monomial structure.

- Solution method: **Polynomial Homotopy Continuation**

Polynomial Homotopy Continuation



Goal:

Regeneration homotopy: more sophisticated approach

Types of Solutions

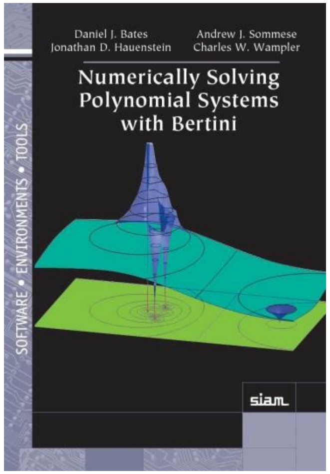
- Polynomial homotopy attempts to find ALL of the roots of a system, including:
  - Roots at infinity
  - Finite roots
  - Nonsingular roots
  - Singular roots

The majority of paths track to these.
Limited by multihomogeneous homotopy.
Discarded quickly by regeneration.
Handled efficiently with projective coordinates.

This is what we desire. In this example,
less than **1%** of **264,241,152** roots track to these.

Discarded quickly by regeneration.

  - Target system solved with **regeneration homotopy**
    - Used the Bertini Homotopy Software
    - 24,822,328 paths tracked
    - **1,521,037** finite, nonsingular solutions found
    - **311** hrs on 256×2.2GHz



Parameter Homotopy

<u>The General Strategy for Solving Families of Polynomial Systems</u>

1. Find all solutions for a numerically general system by any means possible
   - Regeneration homotopy
   - Multihomogeneous homotopy
   - Non-homotopy methods

Computationally expensive:
311 hours for a single solve
Regen tracked 24,822,328 paths
Found 1,521,037 solutions

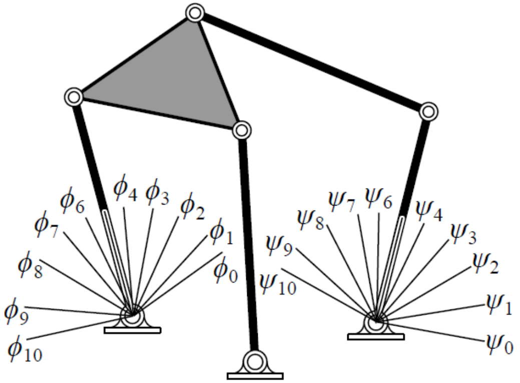
2. Use the results from step 1 as start points for a homotopy that solves a specific system
   - Avoids endpoints at infinity

Computationally efficient:
2 hours per solve
Tracked 1,521,037 paths

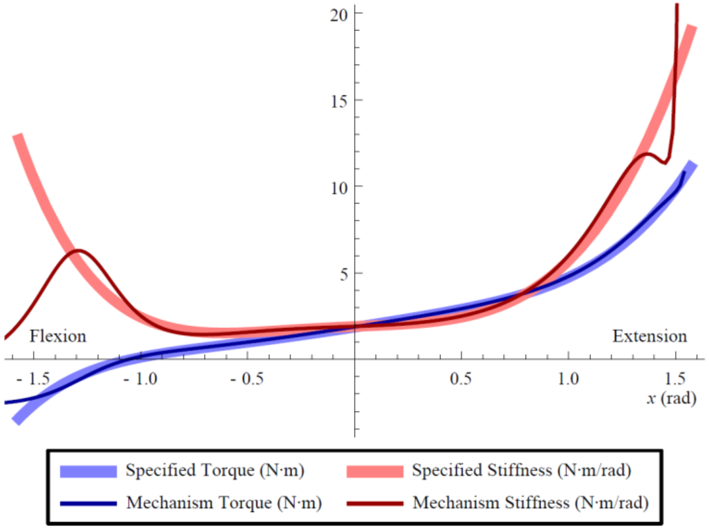
Once a complete solution to a system is found, we can find the solutions to similar systems fast!

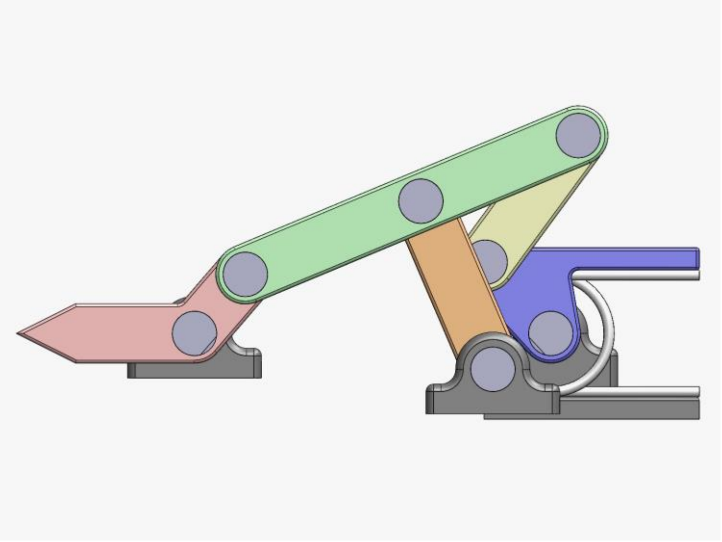
Stephenson III Function Generation

- Stephenson III function generation
  - Degree: 55,050,240 for 11 positions
  - Size of general solution set: 834,441
  - Initial computation: 40 hrs on 512×2.6GHz (multihomogeneous homotopy)
  - Proceeding computations: 50 min on 64×2.2GHz (parameter homotopy)



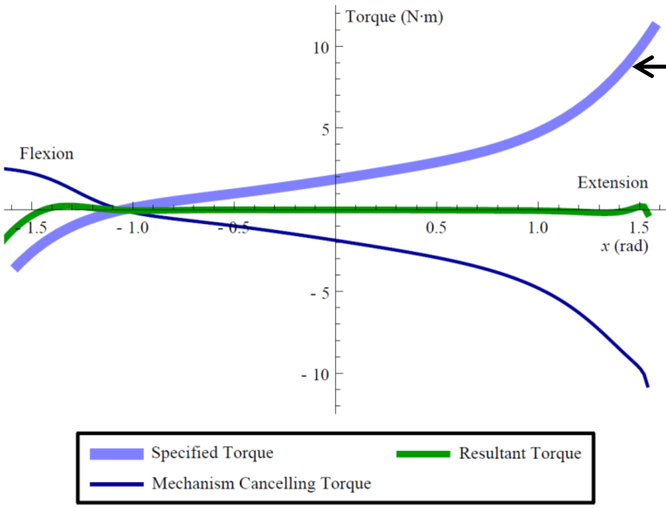
- Design of torque cancelling linkages
  - By placing a linear torsion spring on one end, a function generator can be synthesized to create a specified torque or stiffness profile





Stroke Rehabilitation Application

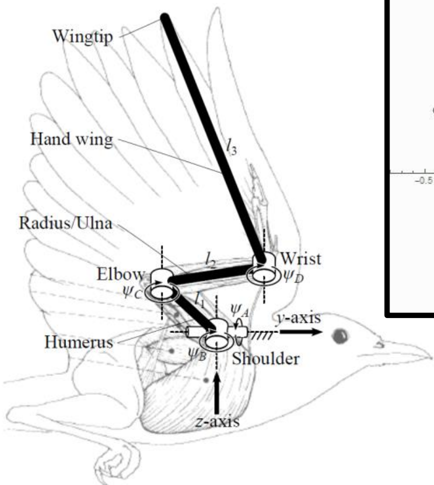
- Applications for torque cancelling include stroke

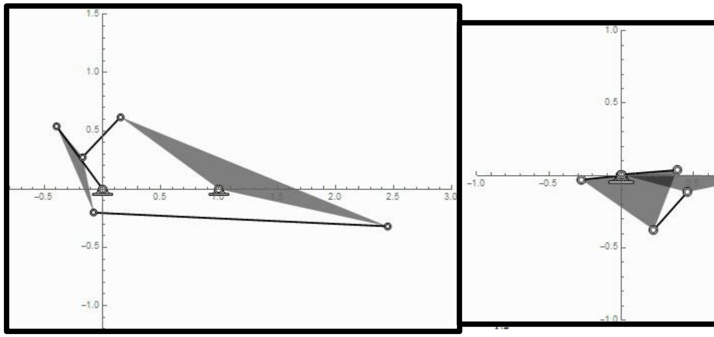


From measurements in stroke survivors' wrists

Results

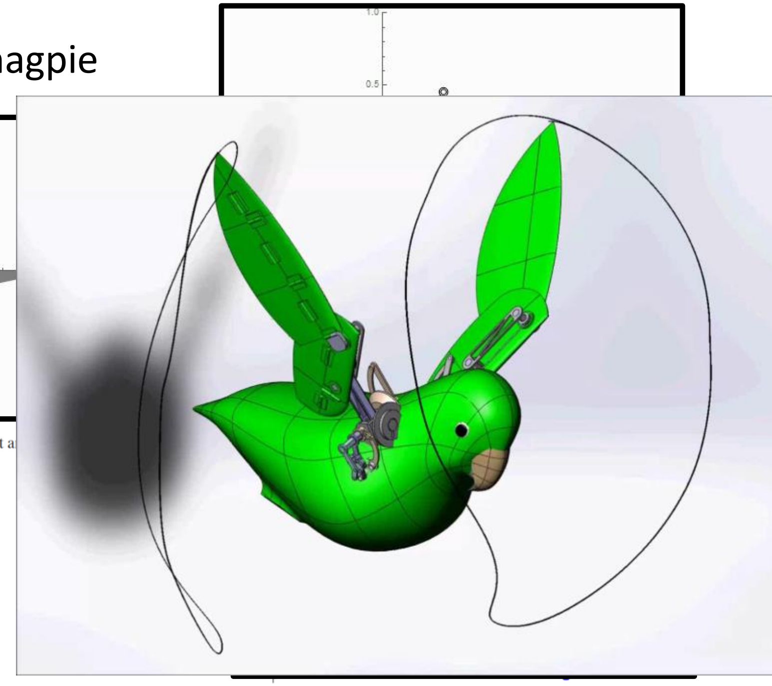
Biomimetic Wing Motion – Joint Angles of the Black-billed magpie



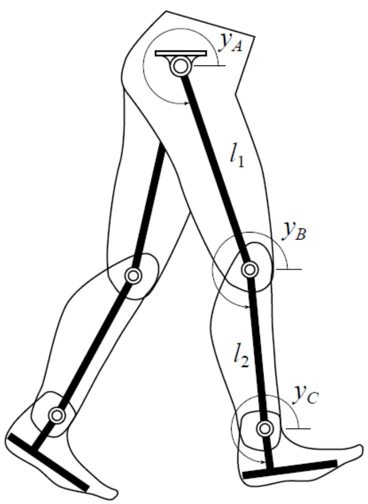


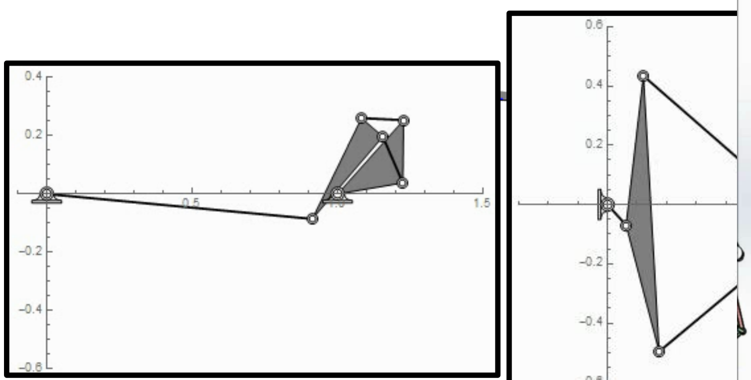
(a) Joint angle ψ_A

(b) Joint a



Biomimetic Human Walking Gait – Planar Joint Angles of Hip, Knee, and Ankle





(a) Hip joint function $\Delta y_A = f_A(t)$

(b) Knee joint f

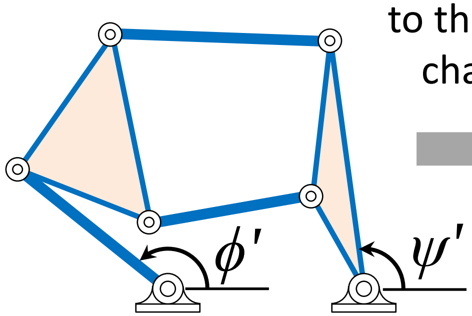
Constrained RR Method

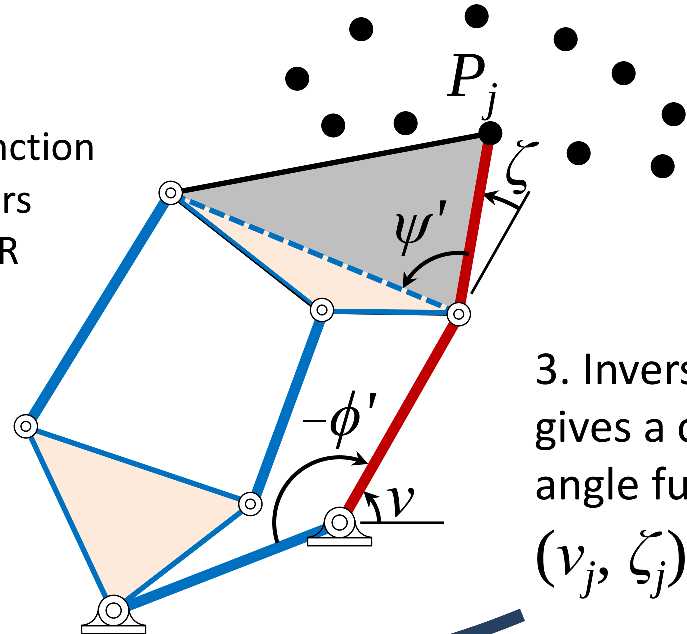
1. Begin by specifying an RR chain

2. Select a set of 11 points to move the RR chain through

The resulting six-bar traces through the 11 points

5. Attach function generators to the RR chain





3. Inverse kinematics gives a coordinated joint angle function

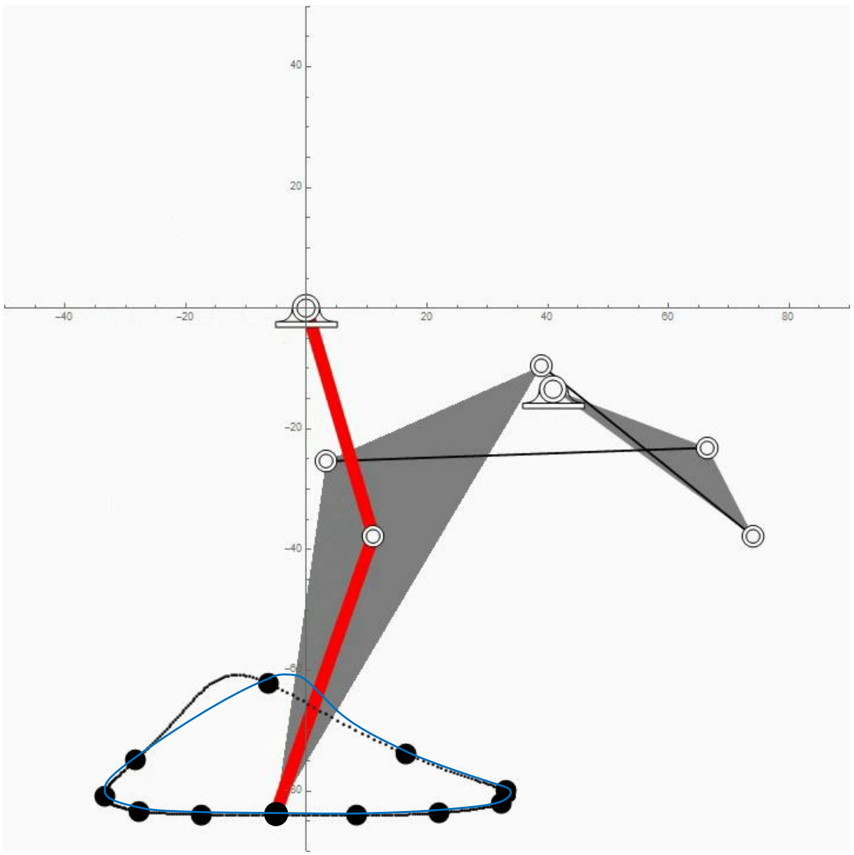
$(\nu_j, \zeta_j), j=0,\ldots,10$

4. Solve for 11 point Stephenson II function generators

Path generation is inverted to function generation

Linkages still need to be verified

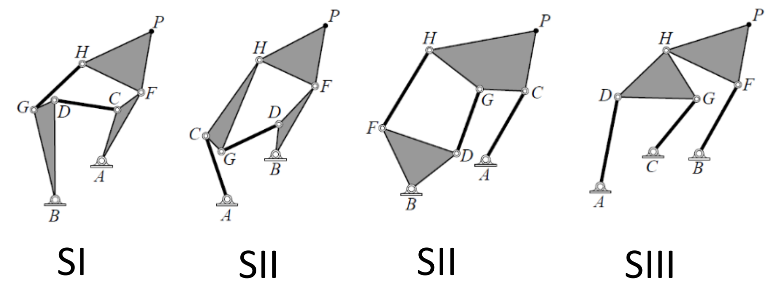
Example

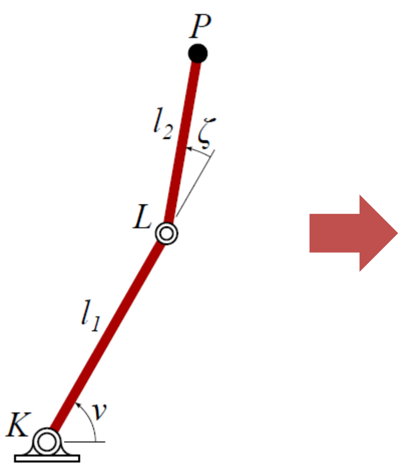


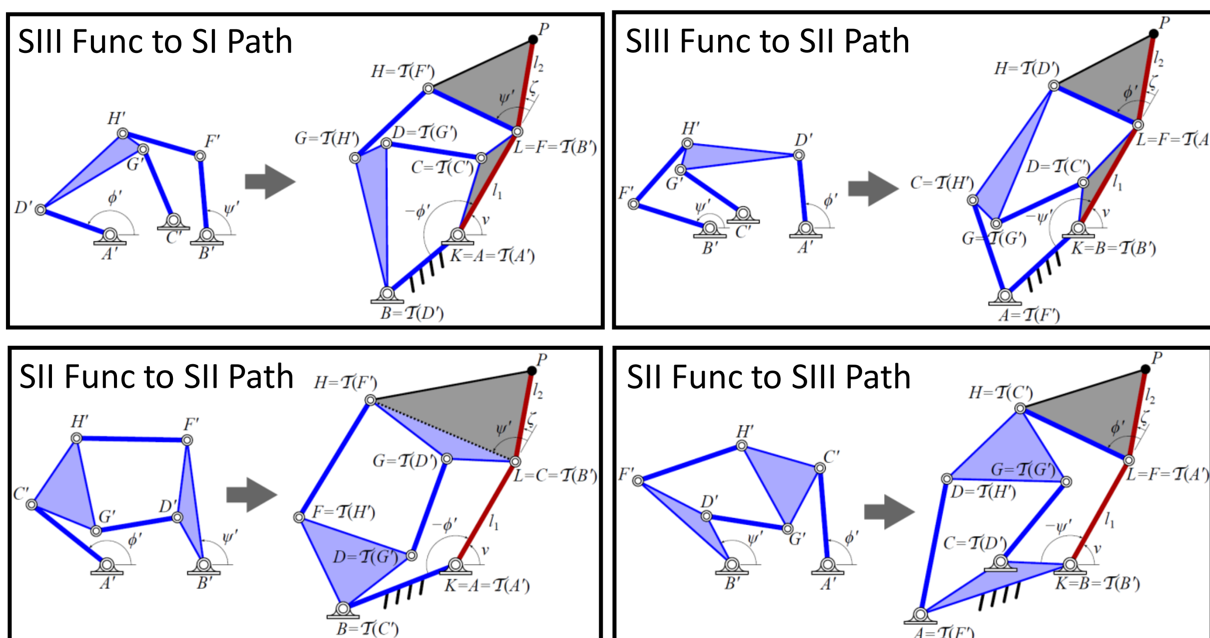
j	x	y
0	−5.160	−83.957
1	8.346	−84.026
2	21.993	−83.632
3	32.259	−82.128
4	33.018	−79.911
5	16.497	−73.889
6	−6.363	−62.120
7	−28.276	−74.865
8	−33.406	−80.964
9	−27.733	−83.440
10	−17.440	−84.032

Stephenson Path Generators

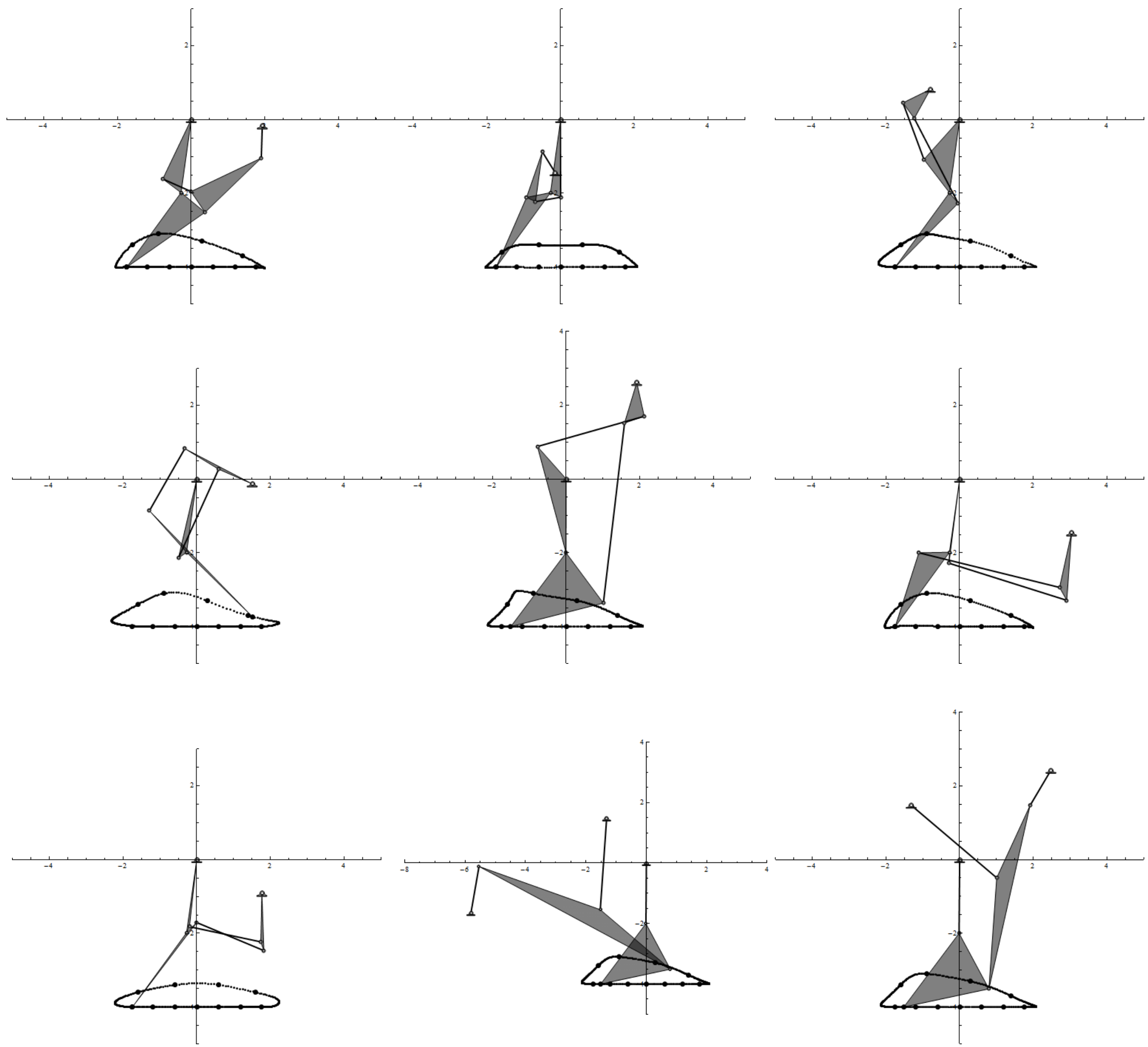
- Goal: Find dimensions of Stephenson linkages so that they move a trace point through 11 points
- Formulated as the synthesis of an RR chain constrain by a Stephenson function generator
- Solve inverse kinematics of RR chain to find joint angles
- Solve for function generators that constrain those joint angles



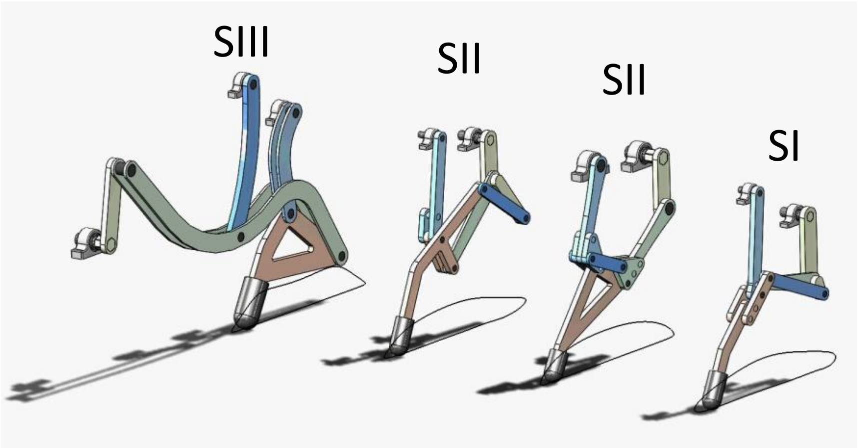


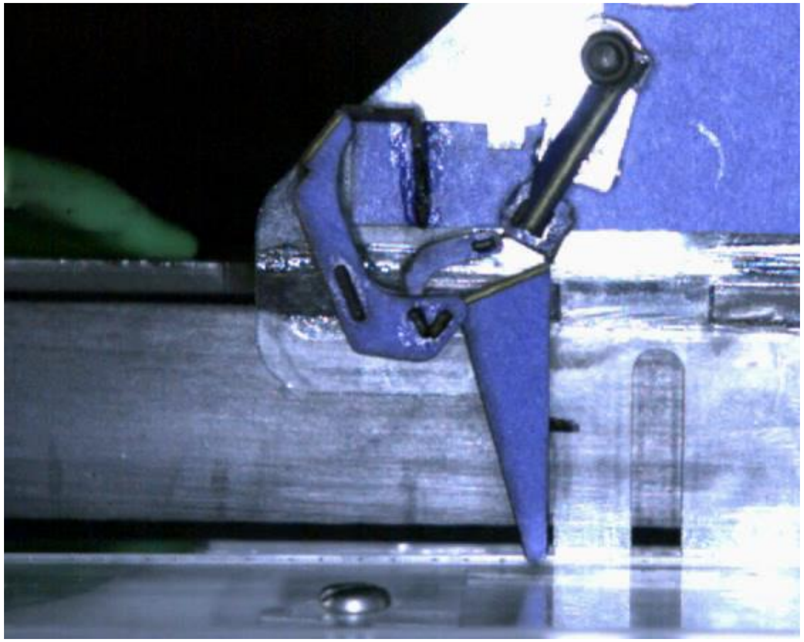


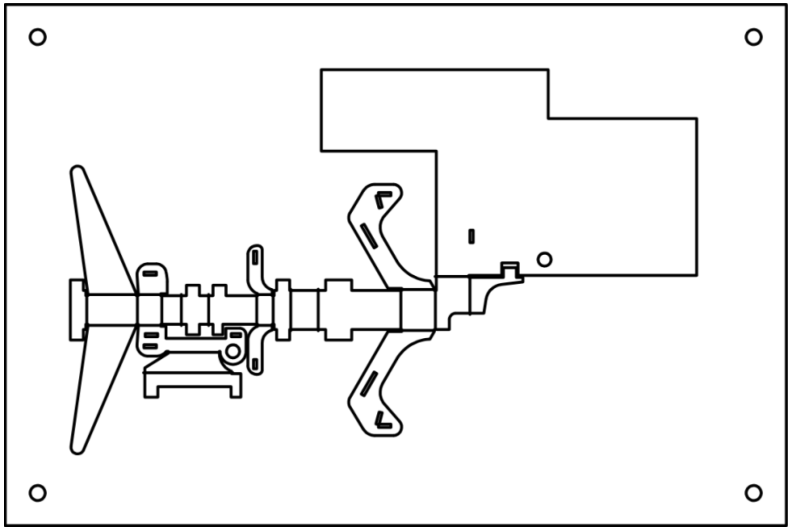
Design Exploration



Exploration of other gaits



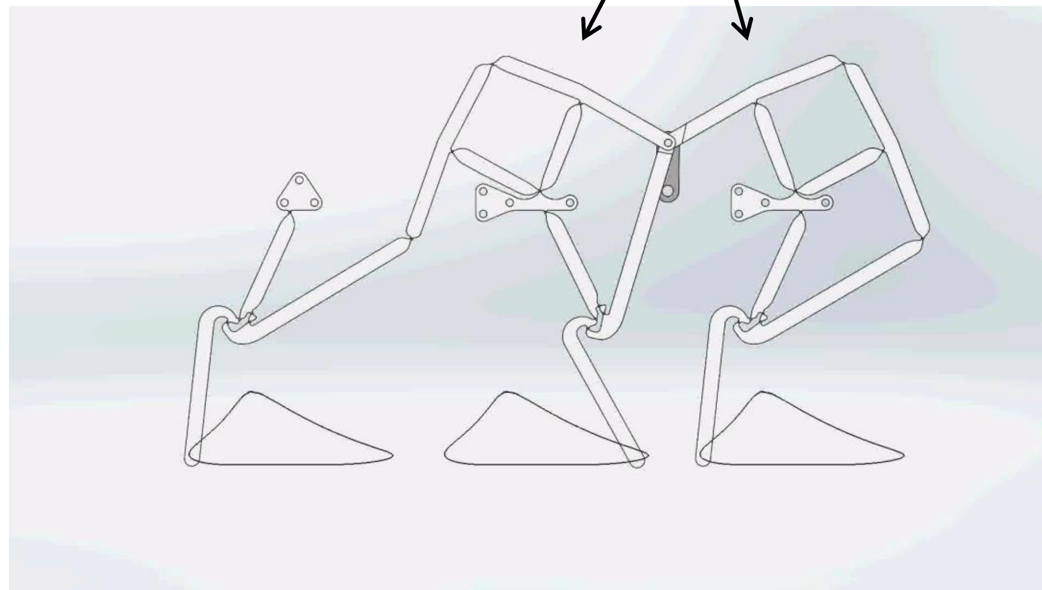




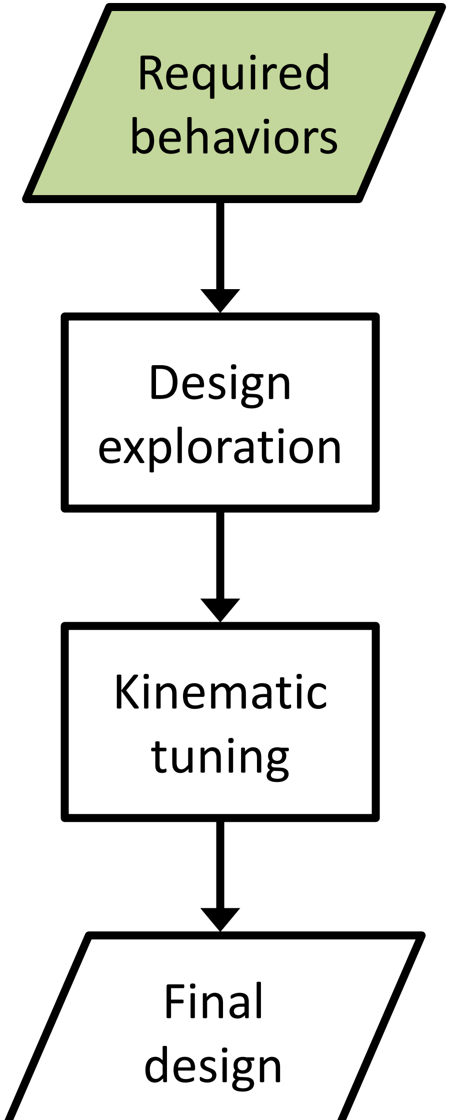
Prototyping a robot

- A leg design was selected and manufactured as a flexure linkage
- Lasercut polypropylene, each leg ¼” x ¼”
- Robot length 30 cm

Pantograph linkages replaces belts



The Design Approach

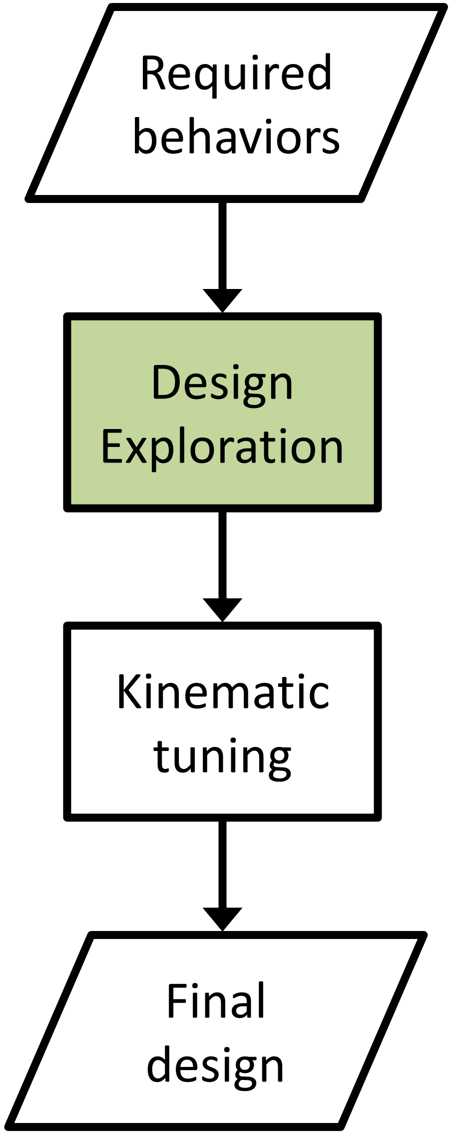


Define Requirements

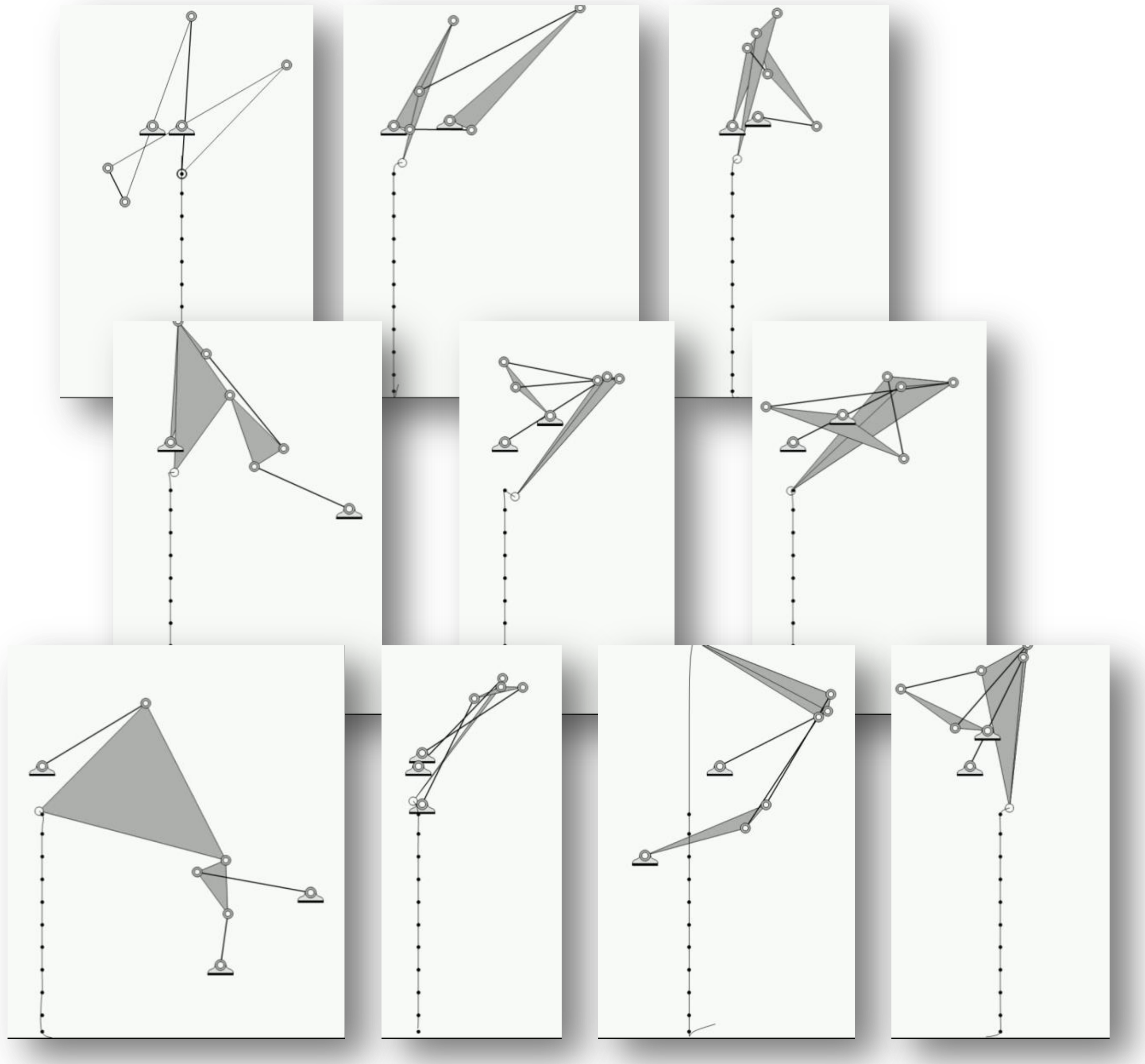
<u>Required Behaviors</u>

1. Traces a straight line
2. Long stroke
3. Input pivot near line-of-action
4. Compact dimensions
5. Input link rotates over large range
6. Low mech. adv. at top of stroke
7. Constant ground reaction force
8. Angular momentum balanced

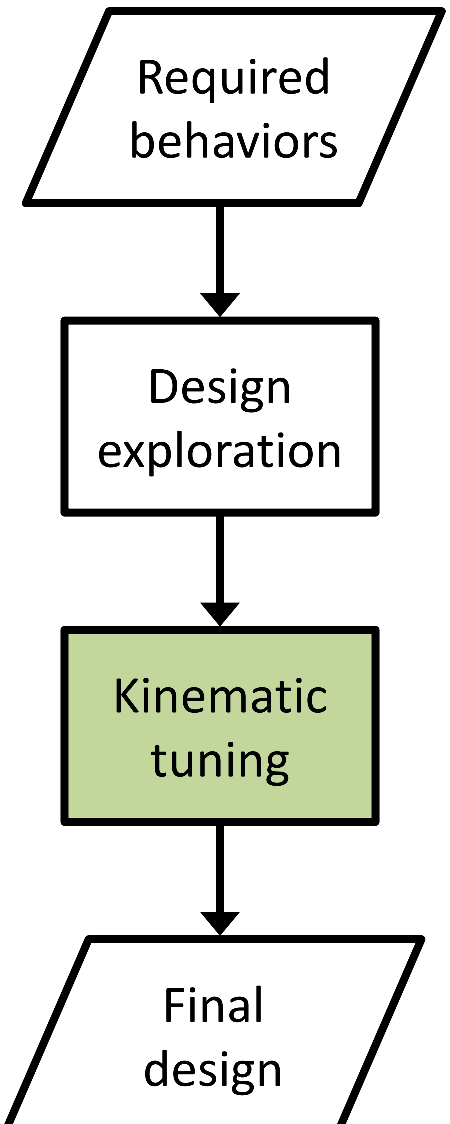
The Design Approach



Generate An Atlas of Designs

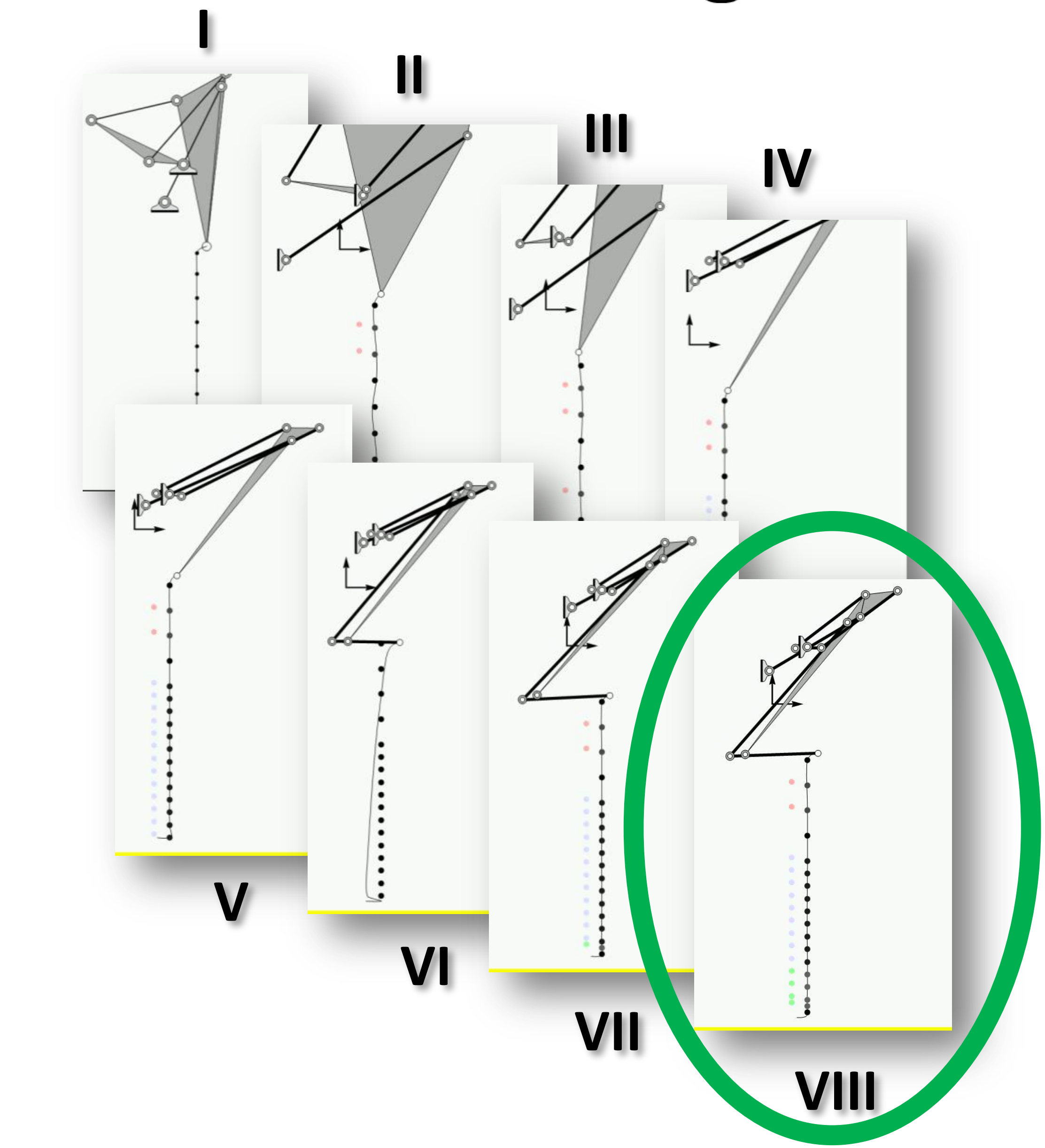


The Design Approach

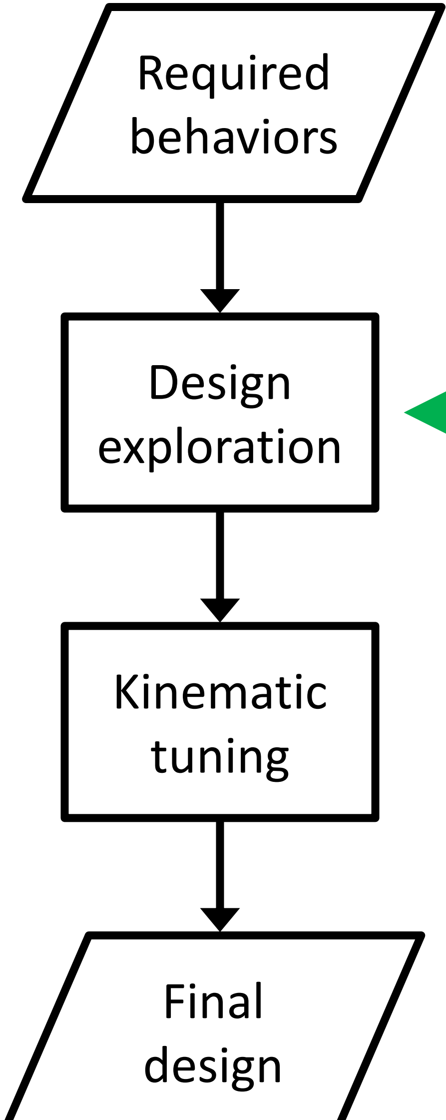


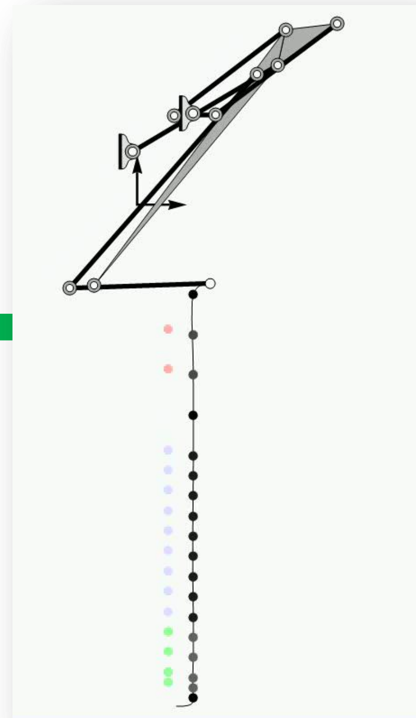
Iterative Design Optimization

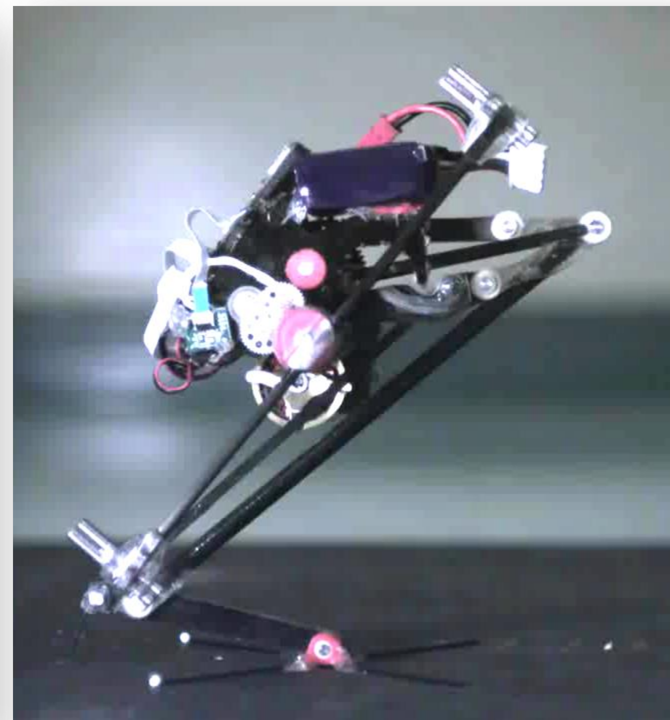
Final Design



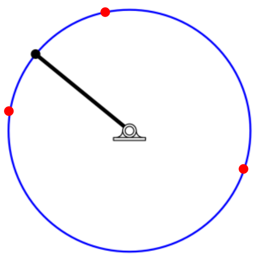
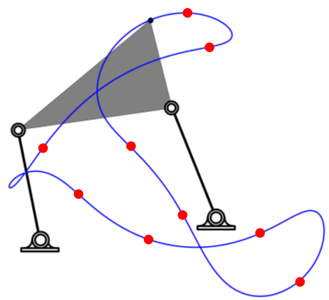
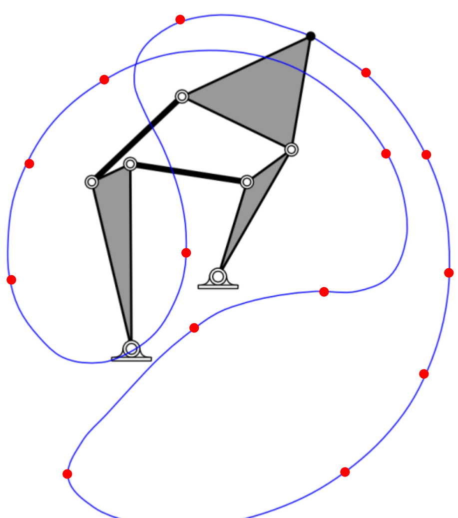
The Design Approach



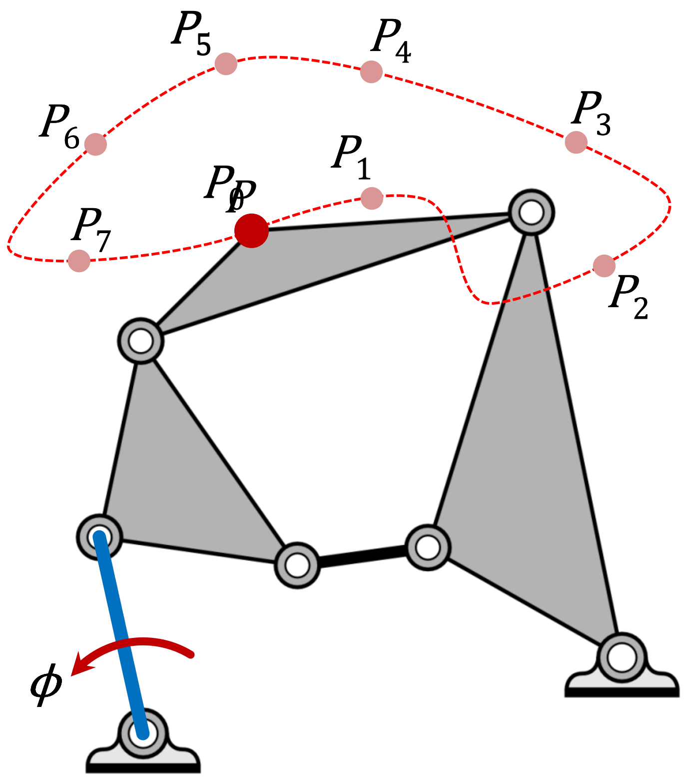




A Simplified History

		No. of Points	No. of Mechanisms	
Crank		3	1	(first discovered in ?)
Four-bar		9	4,326	(first computed in 1992)
Six-bar		15	Unknown >1,000,000	(not yet known)

Stephenson II Timed Curve

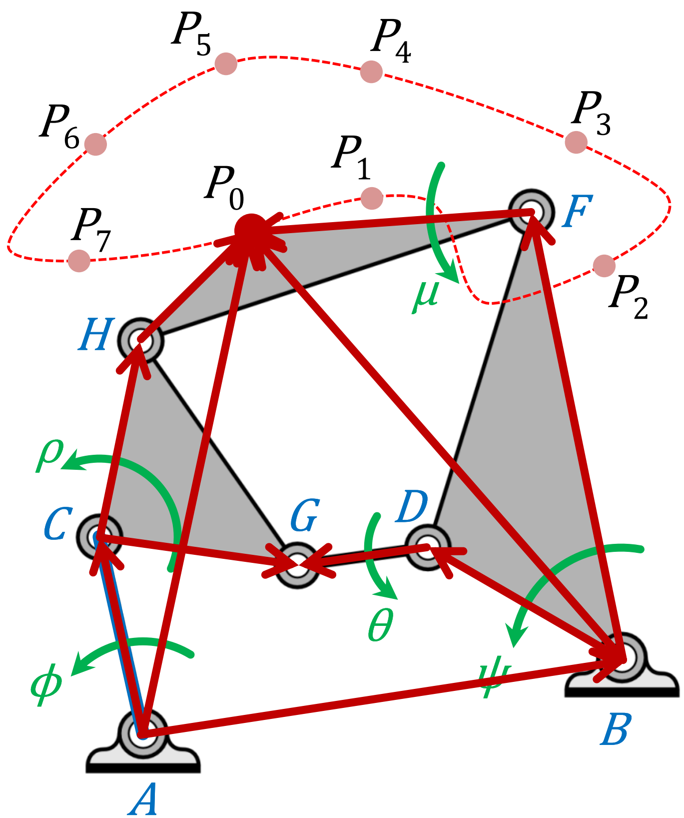


<u>Task</u>

$(0, P_0), (\phi_1, P_1), (\phi_2, P_2), (\phi_3, P_3),$
$(\phi_4, P_4), (\phi_5, P_5), (\phi_6, P_6), (\phi_7, P_7)$

Coordinate input crank with output point

Stephenson II Timed Curve



Joint coordinates

Rotation operators

$Q = e^{i\phi}$ $\quad R = e^{i\rho}$ $\quad S = e^{i\psi}$

$T = e^{i\theta}$ $\quad U = e^{i\mu}$

Loop equations

$$A + Q_j(C - A) + R_j(H - C) + U_j(P_0 - H) = P_j$$

$$B + S_j(F - B) + U_j(P_0 - F) = P_j$$

$$A + Q_j(C - A) + R_j(G - C) - B - S_j(D - B) - T_j(G - D) = 0$$

Stephenson II Timed Curve

Loop equations

$$A + Q_j(C - A) + R_j(H - C) + U_j(P_0 - H) = P_j$$
$$\bar{A} + \bar{Q}_j(\bar{C} - \bar{A}) + \bar{R}_j(\bar{H} - \bar{C}) + \bar{U}_j(\bar{P}_0 - \bar{H}) = \bar{P}_j$$

$$B + S_j(F - B) + U_j(P_0 - F) = P_j$$
$$\bar{B} + \bar{S}_j(\bar{F} - \bar{B}) + \bar{U}_j(\bar{P}_0 - \bar{F}) = \bar{P}_j$$

$$A + Q_j(C - A) + R_j(G - C) - B - S_j(D - B) - T_j(G - D) = 0$$
$$\bar{A} + \bar{Q}_j(\bar{C} - \bar{A}) + \bar{R}_j(\bar{G} - \bar{C}) - \bar{B} - \bar{S}_j(\bar{D} - \bar{B}) - \bar{T}_j(\bar{G} - \bar{D}) = 0$$

Unit rotations

$$R_j\bar{R}_j = 1$$
$$S_j\bar{S}_j = 1$$
$$T_j\bar{T}_j = 1$$
$$U_j\bar{U}_j = 1$$

Extra substitutions

$$a = A\bar{H}$$
$$b = B\bar{F}$$
$$c = (C - A)\bar{H}$$
$$d = \frac{D - B}{F - B}$$
$$g = \frac{G - C}{H - C}$$
$$k = g(P_0 - H) - d(P_0 - F)$$

Several substitutions

Stephenson II Timed Curve

Synthesis Equations

$\beta_j + \bar{\beta}_j - P_j\bar{P}_j - P_j\bar{P}$ **degree = 2** $\quad j = 1, \dots, 7$

$\xi_j + \bar{\xi}_j - P_j\bar{P}_j - P_j\bar{P}$ **degree = 2** $\quad j = 1, \dots, 7$

total degree = $2^7 \times 2^7 \times 4^7 \times 2^8 \times 2^7$
= 8,796,093,022,208

$U_j k\bar{\zeta}_j + \bar{U}_j\bar{k}\zeta_j - \zeta_j\bar{\zeta}_j - k\bar{k} +$
$(g(H - C) + C - d(F$ **degree = 4** $) - B)\big(\bar{g}(\bar{H} - \bar{C}) + \bar{C} - \bar{d}(\bar{F} - \bar{B}) - \bar{B}\big) = 0 \quad j = 1, \dots, 7$

$a -$ **deg=2** $= 0$ $\quad b -$ **deg=2** $= 0$ $\quad c - (C$ **deg=2** $)\bar{H} = 0$ $\quad k -$ **deg=2** $- H) + d(P_0 - F) = 0$

$\bar{a} -$ **deg=2** $= 0$ $\quad \bar{b} -$ **deg=2** $= 0$ $\quad \bar{c} - ($ **deg=2** $\bar{A})H = 0$ $\quad \bar{k} - \bar{g}($ **deg=2** $- \bar{H}) + \bar{d}(\bar{P}_0 - \bar{F}) = 0$

U **deg=2** $1 = 0 \quad j = 1, \dots, 7$

Spoiler Alert! **Approx 1,500,000 finite roots**

Intermediate expressions

$$\beta_j = U_j\big(P_0\big(\bar{P}_j - \bar{A} - \bar{Q}_j(\bar{C} - \bar{A})\big) - \bar{P}_j H + \bar{a} + \bar{Q}_j\bar{c}\big) + Q_j(C - A)\big(\bar{P}_j - \bar{A}\big) + A\big(\bar{P}_j - \bar{C} - \bar{A}\big) + H(\bar{P}_0 - \bar{C})$$

$$\xi_j = U_j\big(P_0\big(\bar{P}_j - \bar{B}\big) - \bar{P}_j F + \bar{b}\big) + P_j\bar{B} + P_0\bar{F} - b$$

$$\zeta_j = A - B + Q_j(C - A) + g\big(P_j - A - Q_j(C - A)\big) - d(P_j - B)$$

8,796,093,022,208

Why the discrepancy?

1,500,000

Sparse System

Start system

$$(a_1x + a_2y + 1)(a_3x + a_4y + 1)(a_5x + a_6y + 1) = 0$$

$$(a_7x + a_8y + 1)(a_9x + a_{10}y + 1)(a_{11}x + a_{12}y + 1) = 0$$

No. of roots: 9

Monomials: $\{x^3, y^3, x^2y, xy^2, x^2, y^2, xy, x, y, 1\}$

Target system

$$c_1x^3 + c_2xy + c_3y + 1 = 0$$

$$c_4x^3 + c_5xy + c_6y + 1 = 0$$

No. of roots: 4

Monomials: $\{x^3, xy, y, 1\}$

Expanded form:

$$b_1x^3 + b_2y^3 + b_3x^2y + b_4xy^2 + b_5x^2 + b_6y^2 + b_7xy + b_8x + b_9y + 1 = 0$$

$$b_{10}x^3 + b_{11}y^3 + b_{12}x^2y + b_{13}xy^2 + b_{14}x^2 + b_{15}y^2 + b_{16}xy + b_{17}x + b_{18}y + 1 = 0$$

** a & c coefficients are generic complex numbers

Start	Homotopy	Target
b_1	$\longrightarrow$	c_1
b_2	$\longrightarrow$	0
b_3	$\longrightarrow$	0
b_4	$\longrightarrow$	0
b_5	$\longrightarrow$	0
b_6	$\longrightarrow$	0
b_7	$\longrightarrow$	c_2
b_8	$\longrightarrow$	0
b_9	$\longrightarrow$	c_3

Sparse System

Start system

$$(a_1x + a_2y + 1)(a_3x + a_4y + 1)(a_5x + a_6y + 1) = 0$$

$$(a_7x + a_8y + 1)(a_9x + a_{10}y + 1)(a_{11}x + a_{12}y + 1) = 0$$

No. of roots: 9

Monomials: $\{x^3, y^3, x^2y, xy^2, x^2, y^2, xy, x, y, 1\}$

A start system with monomials that match the target would be nice!

Target system

$$c_1x^3 + c_2xy + c_3y + 1 = 0$$

$$c_4x^3 + c_5xy + c_6y + 1 = 0$$

No. of roots: 4

Monomials: $\{x^3, xy, y, 1\}$

Recall Stephenson II example…

Start system

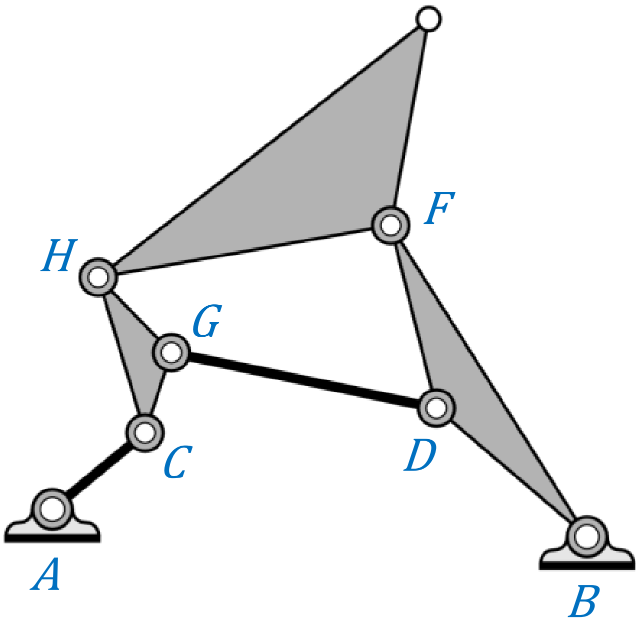
No. of roots: 8,796,093,022,208 $\longrightarrow$

Target system

No. of roots: 1,500,000

Random Startpoints

A randomly generated mechanism…



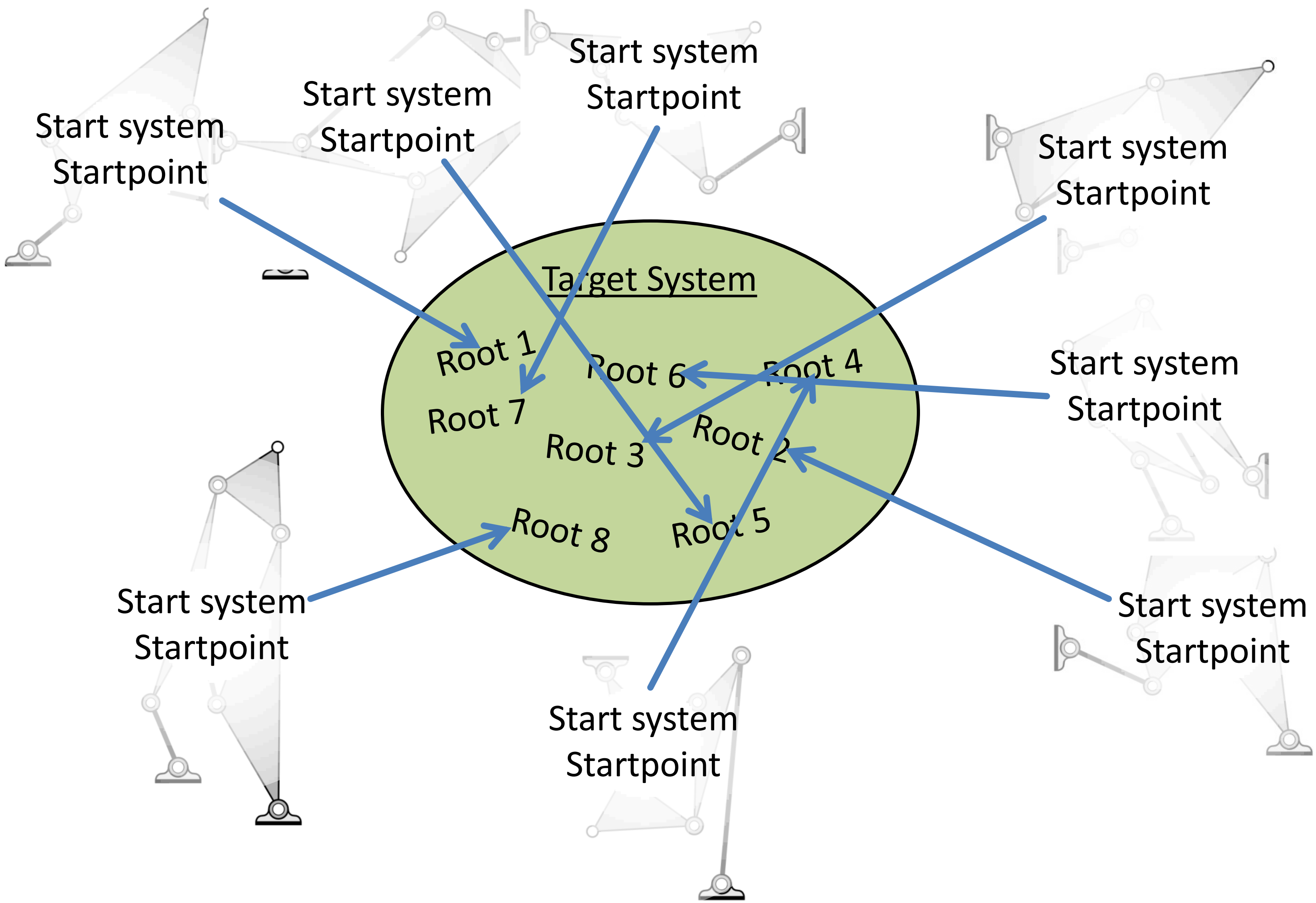
Its movement: Loop equations

Construct a <u>start system</u> with exactly the right monomials

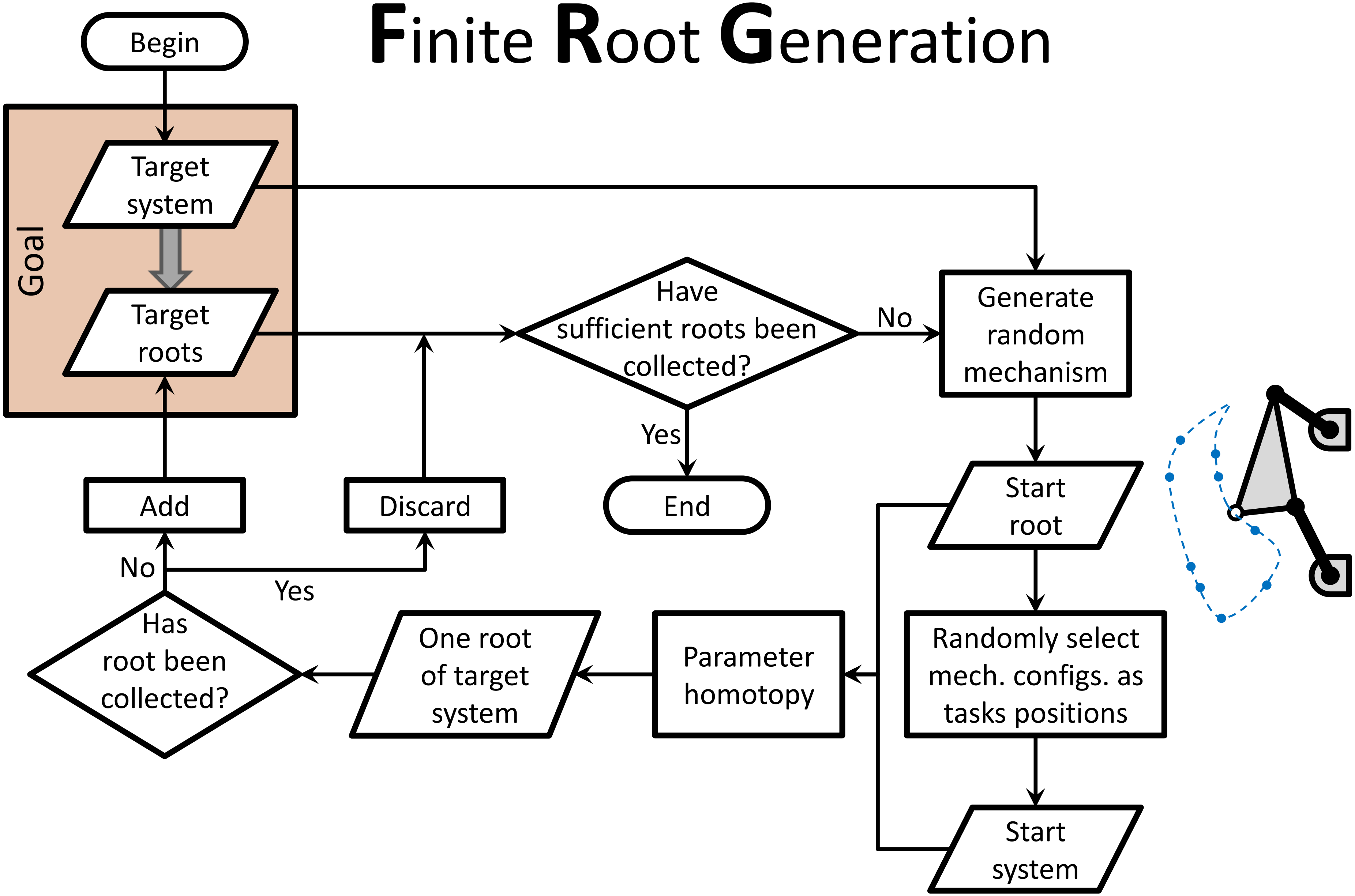
Its dimensions: $A \quad B \quad C \quad D \quad F \quad G \quad H$

Provide a single solution to <u>start system</u>

Random Startpoints



Finite Root Generation



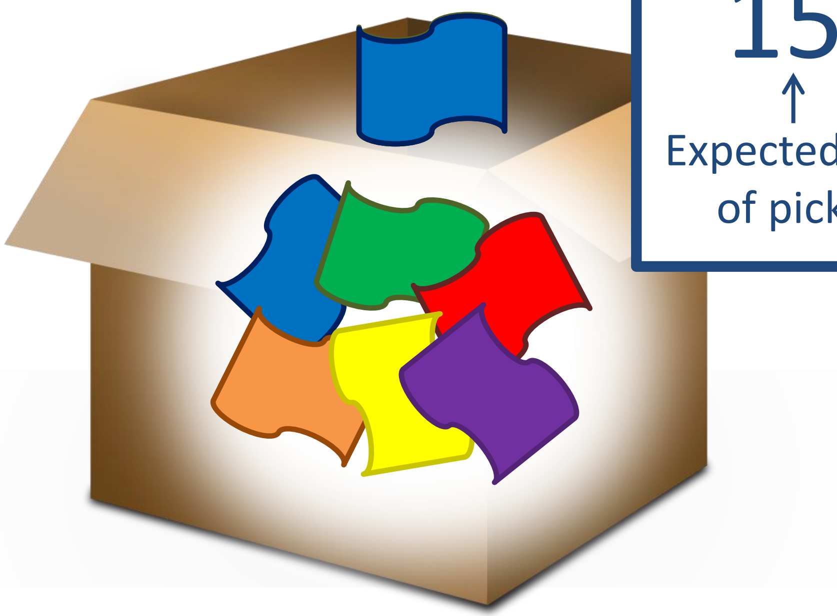
Collecting Coupons

- The process of accumulating roots through FRG is analogous to randomly picking coupons out of a box.
- There are 6 unique different colored coupons in the box

How many picks until all coupons have been collected once?

Probability of picking a new color:

50%



15 ± 6.2

Expected no. of picks

Standard deviation

Red	
Orange	✓
Yellow	
Green	✓
Blue	✓
Violet	

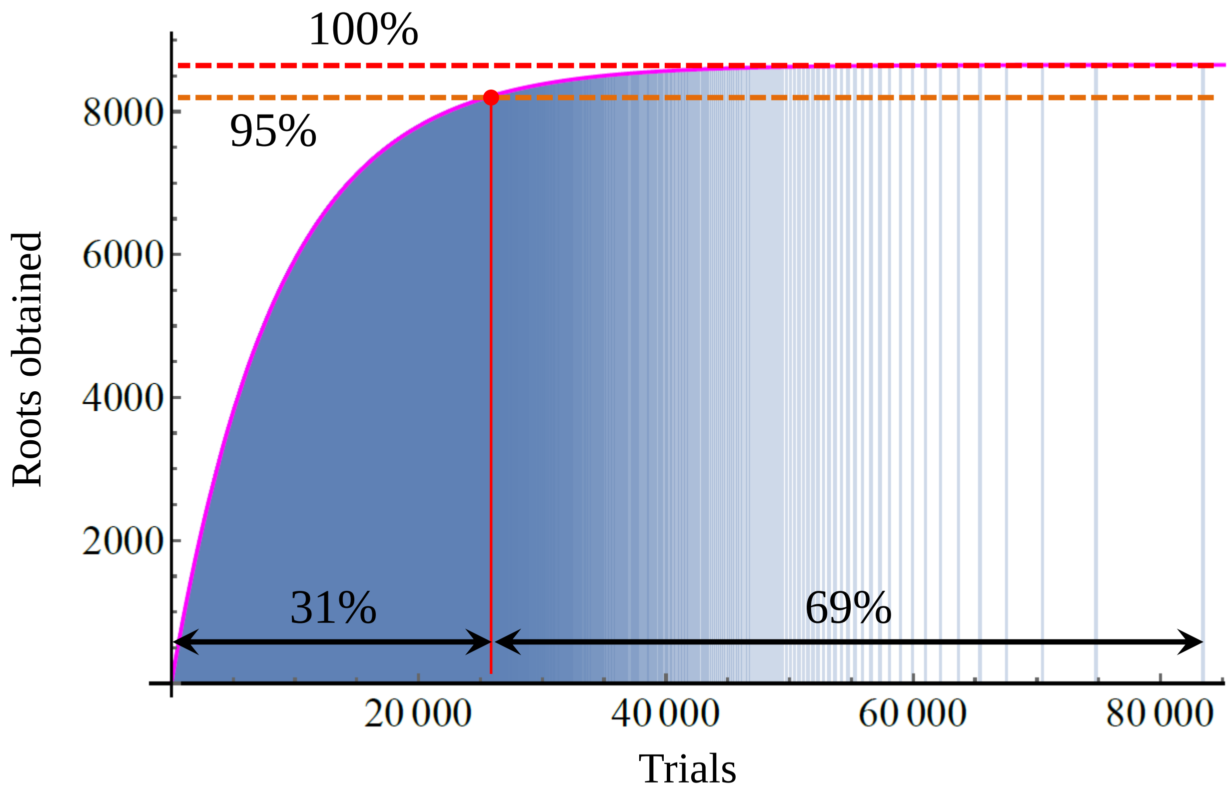
FRG Root Collection

Expected no. of trials to obtain n of N roots

$$T_n = N(H_N - H_{N-n})$$

Harmonic numbers

$$H_N = \sum_{k=1}^{N} \frac{1}{k}$$



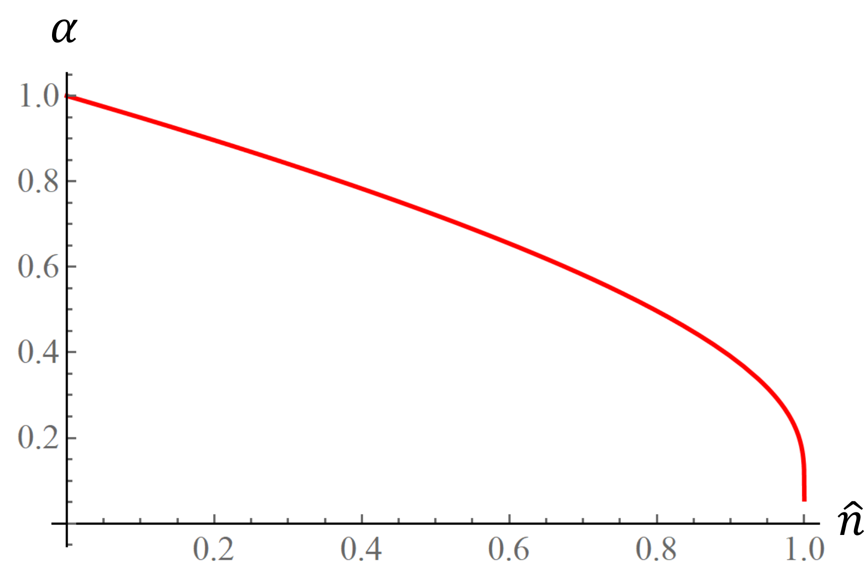
FRG Estimation

<u>Coupon collector model</u>

$$T_n = N(H_N - H_{N-n})$$

Expected no. of trials

Total no. of roots



<u>Approximate coupon collector model</u>

$$T_n \approx N \ln\left(\frac{N}{N-n}\right)$$

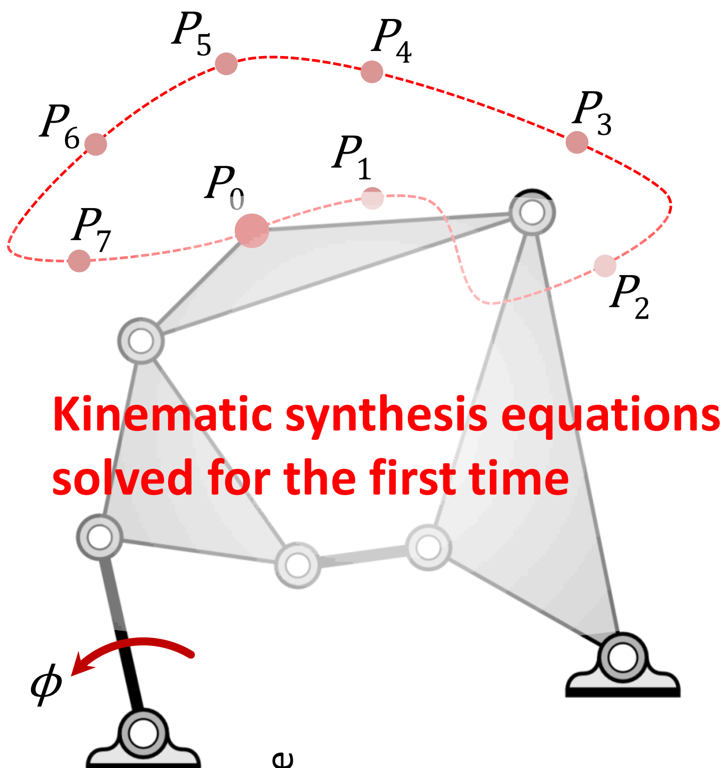
<u>Estimation equation</u>

Percentage of roots collected $\hat{n} = \frac{n}{N}$

New root success rate $\alpha = \frac{n}{T_n}$

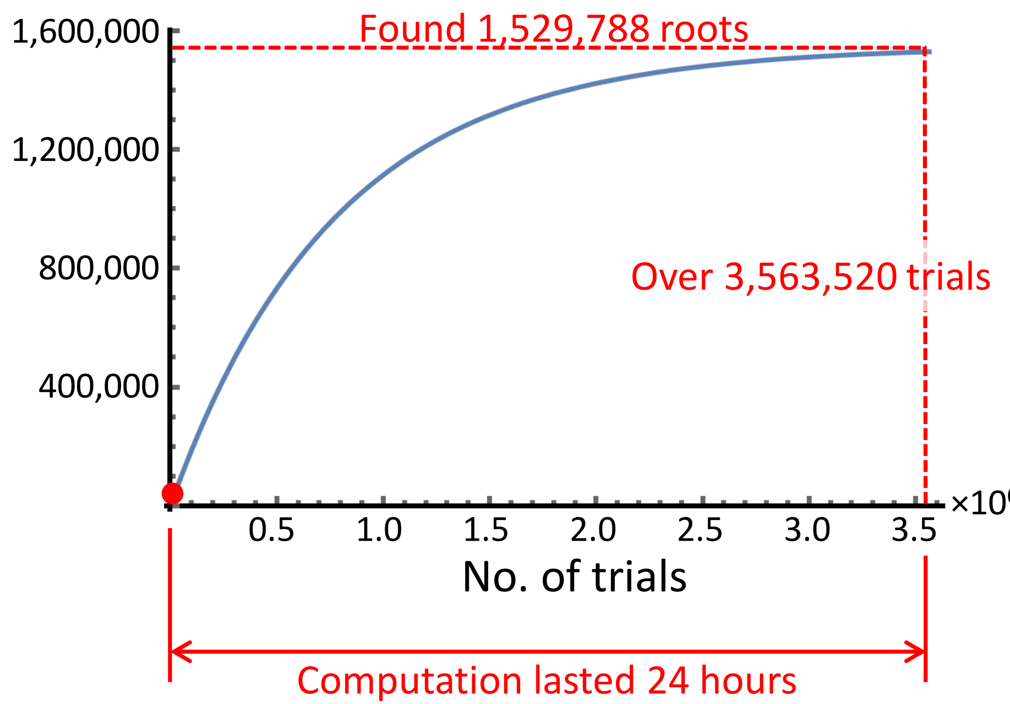
$$\alpha = -\frac{\hat{n}}{\ln(1-\hat{n})}$$

Stephenson II Timed Curve

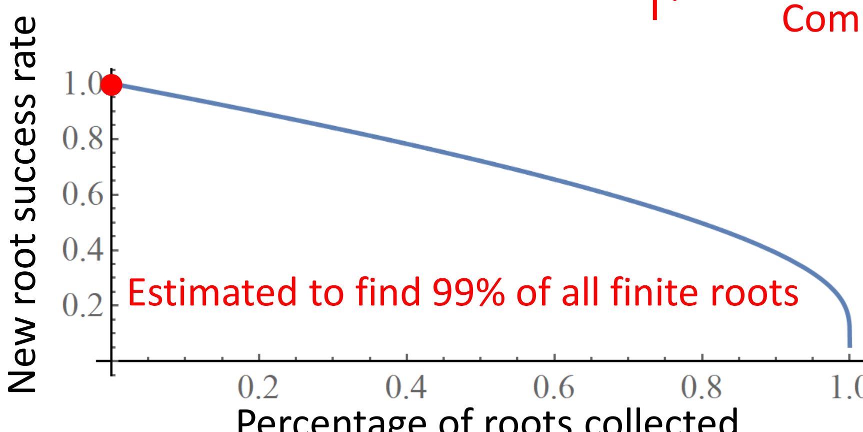


Kinematic synthesis equations solved for the first time

Roots collected*

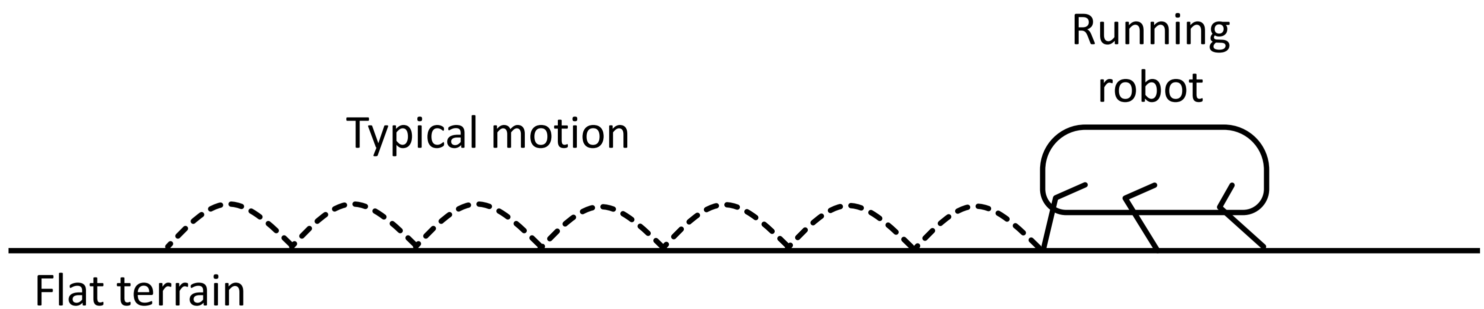


Using a laptop GPU

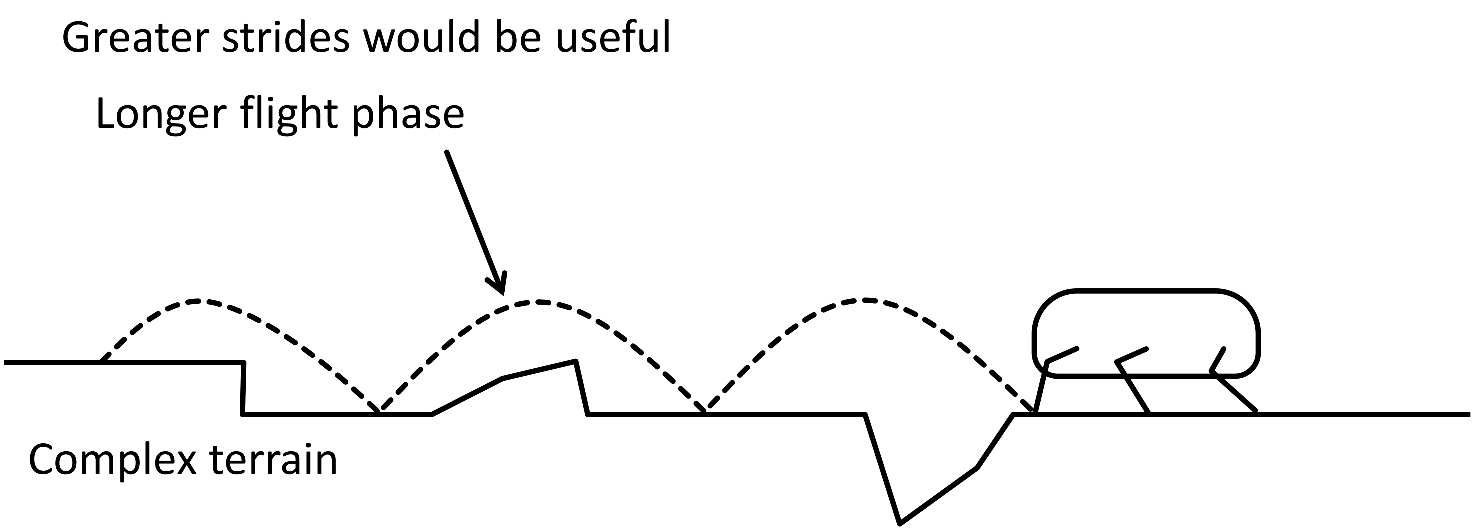


*Cognate structure reduced tracking requirements 50%

Application

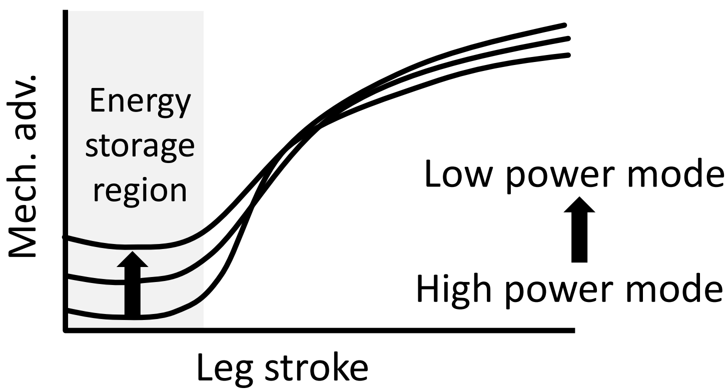


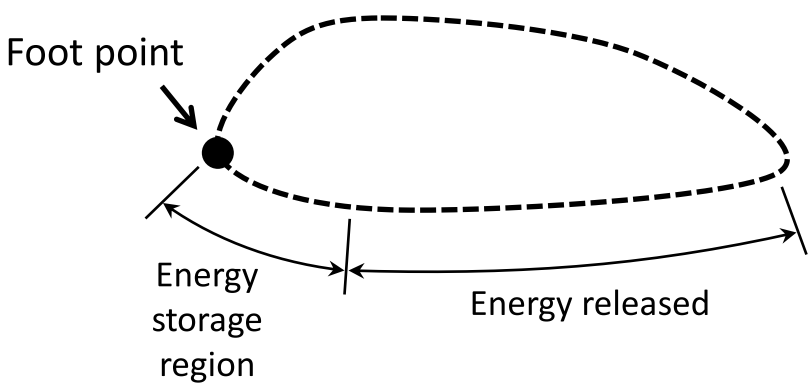
Application

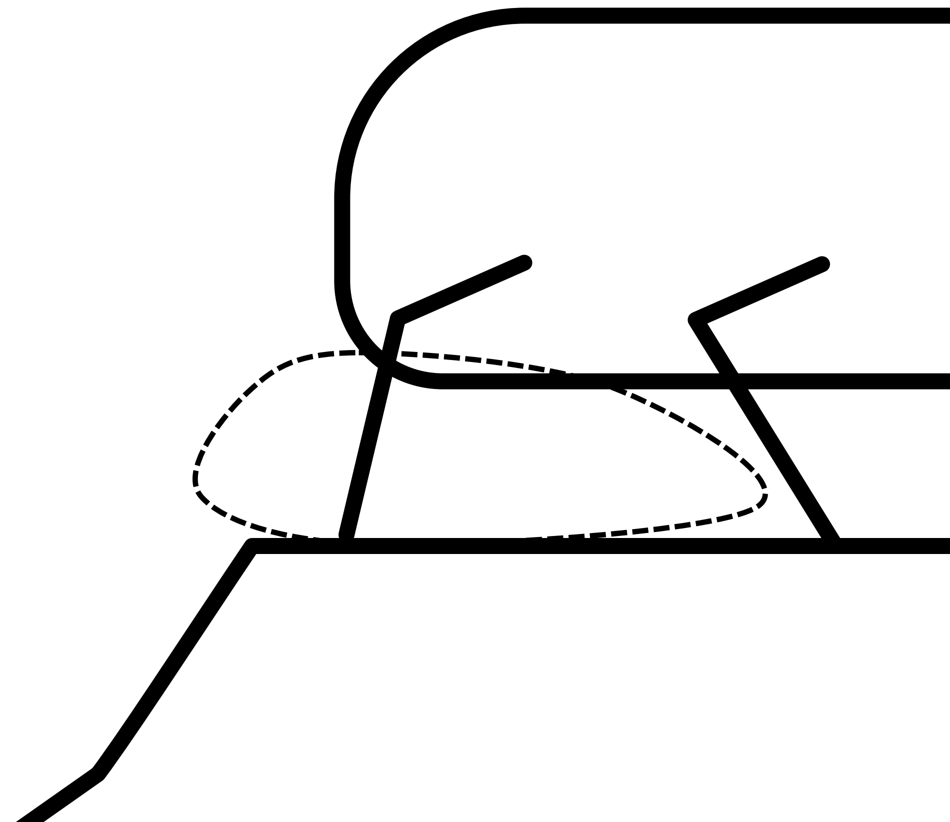


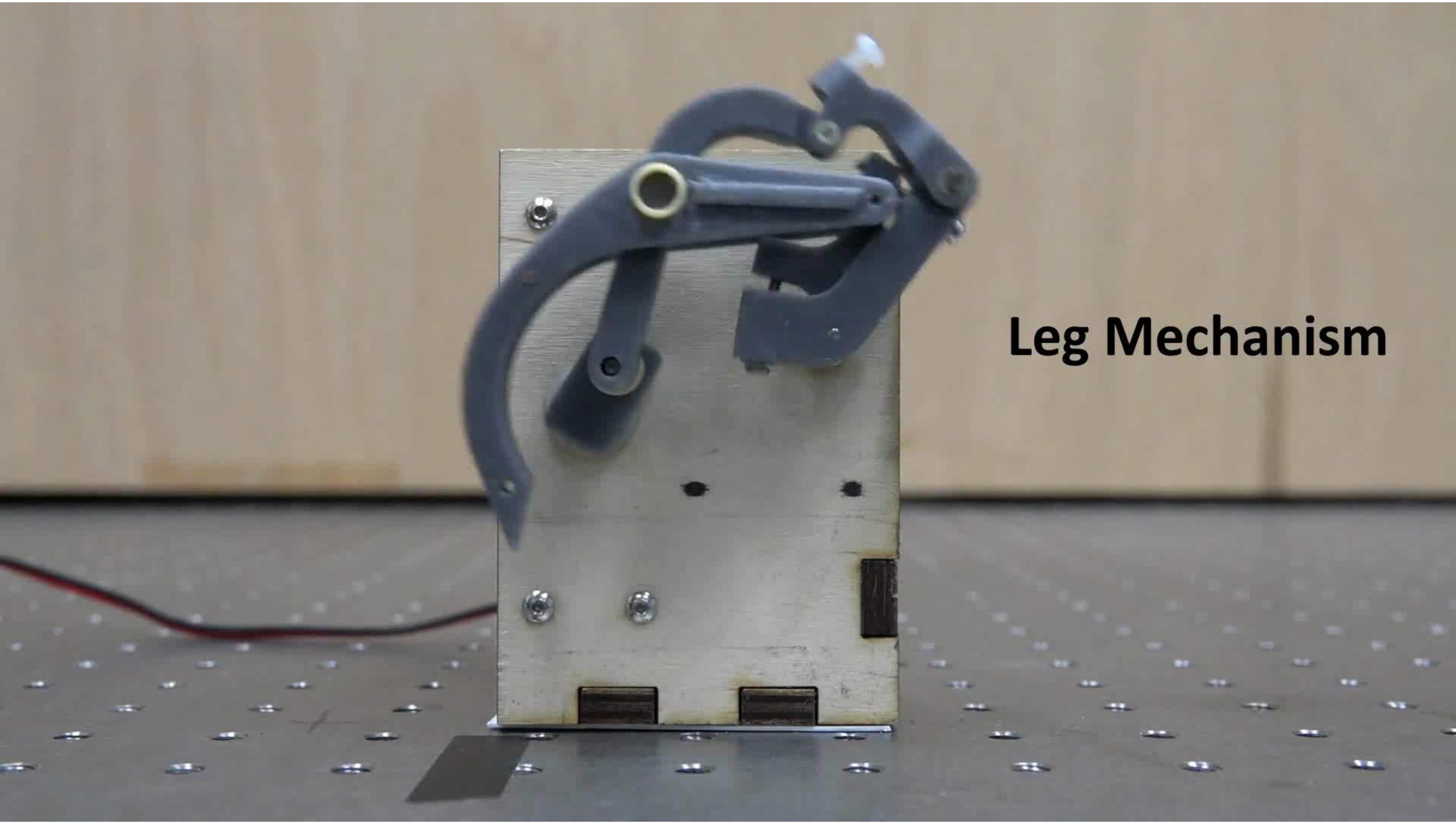
Design requirements for running:

- Cyclic motion
- Special mechanical advantage that pairs with an external spring
- Extra feature: Mech. adv. adjustability



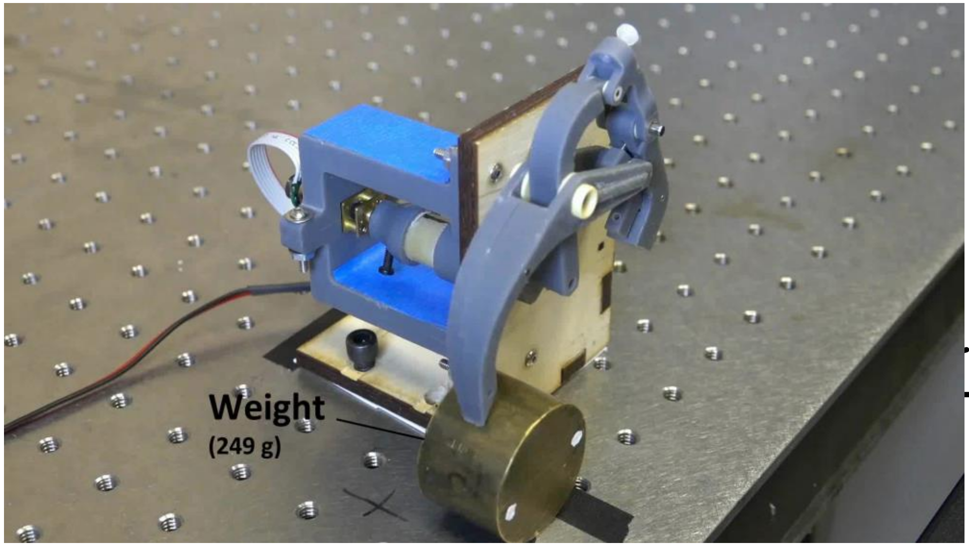


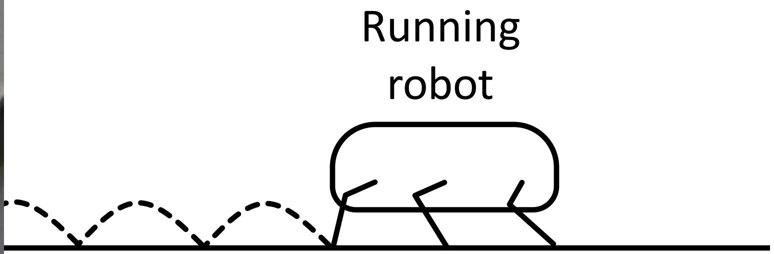


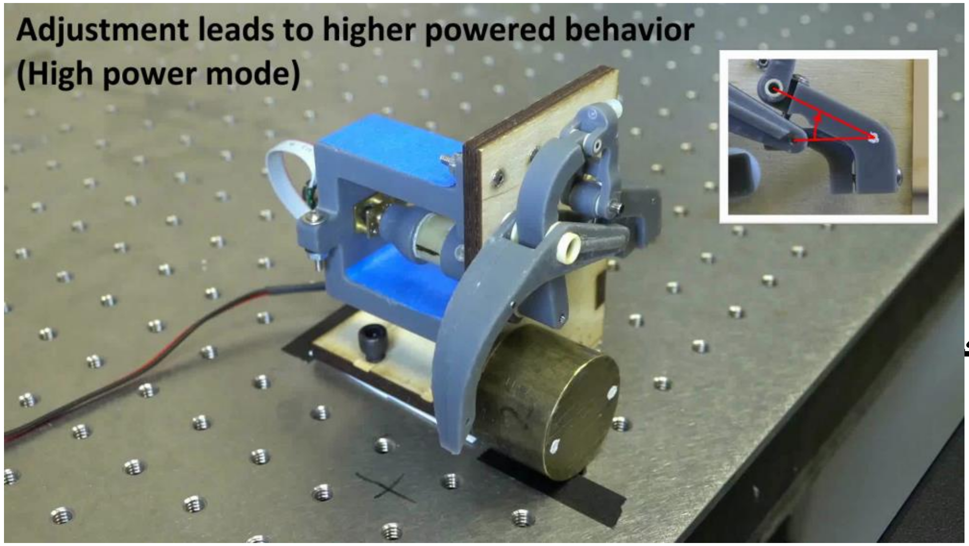


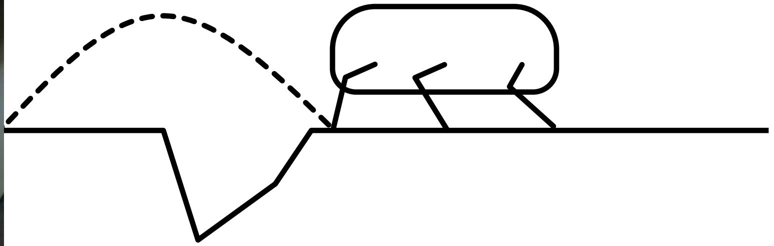
Leg Mechanism

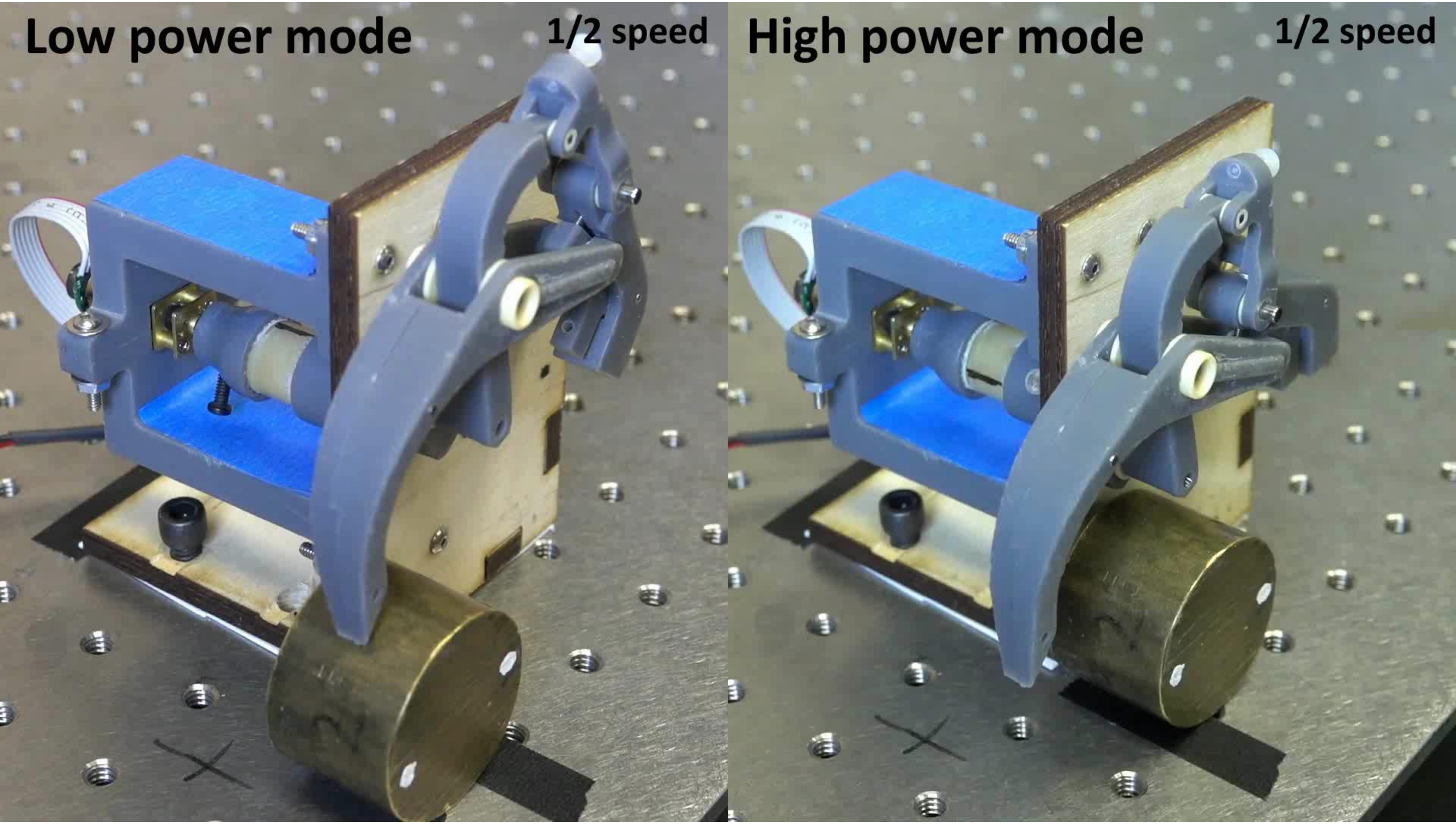
Design Work Performed With This Result

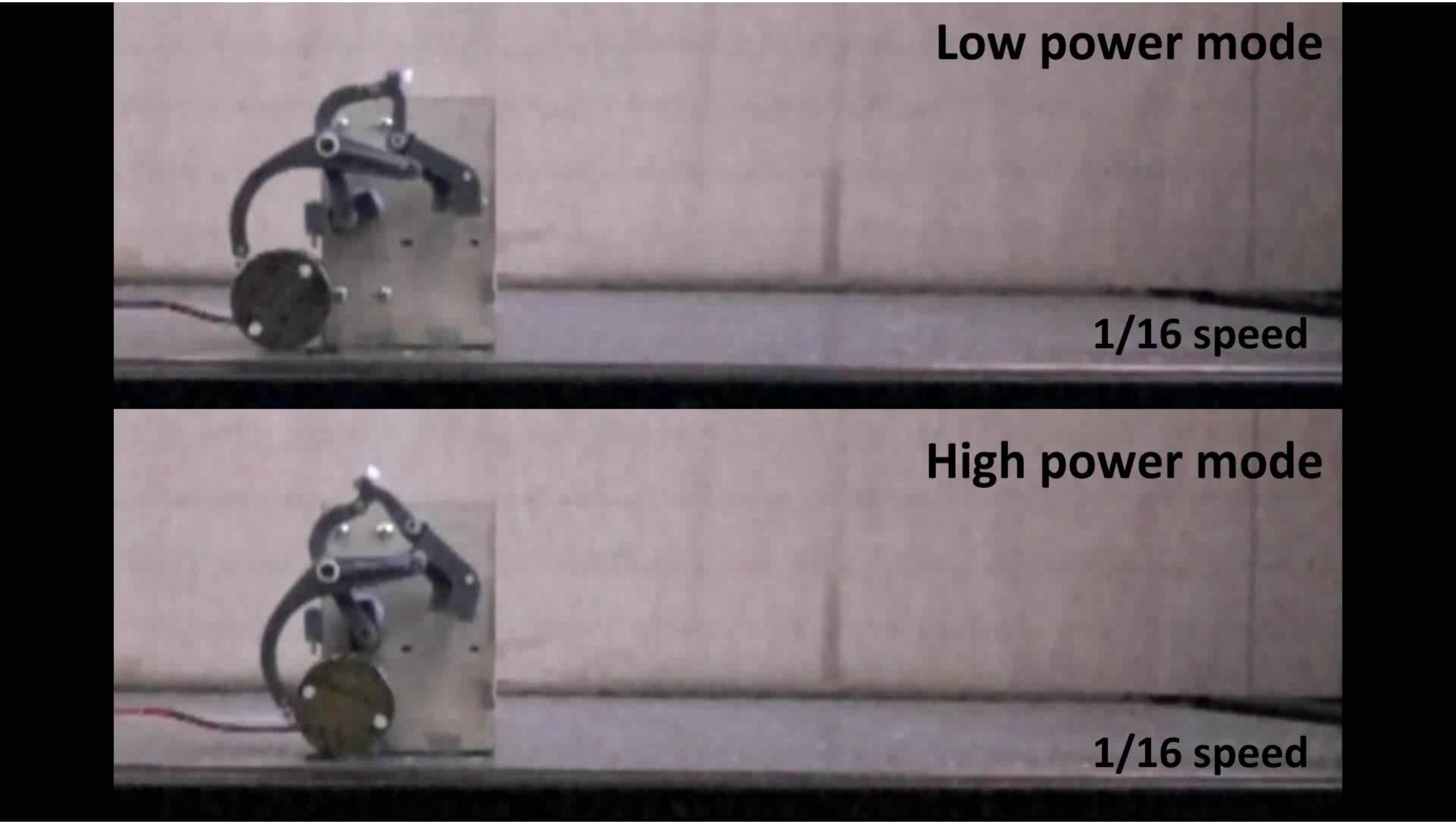









Low power mode
1/2 speed
High power mode
1/2 speed



Low power mode
1/16 speed
High power mode
1/16 speed

Wrap Up

- Homotopy solvers (**Bertini**) allow design space exploration for mechanisms
- Stochastically generating startpoints with certain properties can save a lot on computation
- **F**inite **R**oot **G**eneration scales approximately linearly by the actual number of finite roots
  (essentially exploiting sparse monomial structures)
- Many six-bar design problems still unsolved
  (but they are being zeroed in on)

Thank you!